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Higher order mode analysis of circular coaxial waveguides using finite difference

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This paper presents a simple technique to evaluate the cut-off wavelengths of circular coaxial waveguides employing the method of finite difference. Curvilinear rectangular meshing has been employed to subdivide the cross-sectional region between the two conductors. Helmholtz equation in polar form is solved with the appropriate boundary condition to evaluate the cut-off wavenumbers and correspondingly the cut-off wavelengths for TE and TM modes. Plots of cut-off wavelengths of few modes for various ratios of outer-to-inner radii are depicted. The data obtained from analytical expressions are compared to justify the validity of the analysis.

Keywords: Circular coaxial waveguide, Finite difference, Cut-off wavelengths
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1 Introduction

Coaxial waveguides and the determination of their cut-off frequencies were discussed by Marcuvitz[1]. This involves finding zeros of a function that involves products of Bessel functions of 1st and 2nd kinds for TM modes and products of derivatives of Bessel functions of 1st and 2nd kinds for TE modes. These zeros pertain to a certain order and needs a number of iterations to be performed to obtain a set of cut-off wavenumbers. Finite difference methods in the conventional form[2] have been applied to a variety of cross-sections. However, the same technique involving formation of rectangular meshes to a circular coaxial waveguide does not seem to be appealing in the context of selection of truncation boundaries and appropriate adjacent node points for the region between the inner and outer conductors. The increase in the number of spurious modes generated and the decrease in accuracy are also detrimental to this choice of meshing technique.

In this paper, the curvilinear rectangular grid formation[3] is used to mesh the region between the two conductors, which enables one to encompass the cross-sectional region completely, thus eliminating the necessity of truncation of boundary. This also yields much accurate result as compared to the conventional rectangular grid formation and is much simpler than the cumbersome method of finding roots from the closed form expressions. The Laplacian operator appearing in the Helmholtz equation in polar form is represented as a five-point difference operator along the radial and circumferential directions. The eigen values, which give cut-off frequencies, are evaluated from the characteristic equation expressed in terms of matrices obtained from a set of simultaneous equations satisfying the boundary conditions. A comparison of the numerical data with those evaluated by root finding of expressions[1] is presented. Plots of normalized cut-off wavelength versus various outer-to-inner radii for TM modes have been found to be in excellent agreement with the earlier studies[4]. A similar plot for TE modes is also depicted.

2 Method of finite difference

Consider a circular coaxial waveguide having inner radius r_1 and outer radius r_2 as shown in Fig.1 (a). For the application of the method of finite difference, the entire region between the two circular boundaries is covered by a number of concentric circles and radial lines. The points of intersection of these lines are called nodes. As a result of such sub-division, the entire region between the two concentric circles consists of curvilinear rectangular grid. The Helmholtz equation in polar form is expressed as

$$\frac{1}{r}\frac{\partial U}{\partial r}+\frac{\partial^2 U}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 U}{\partial \Phi^2}+k^2U=0 \qquad ...(1)$$

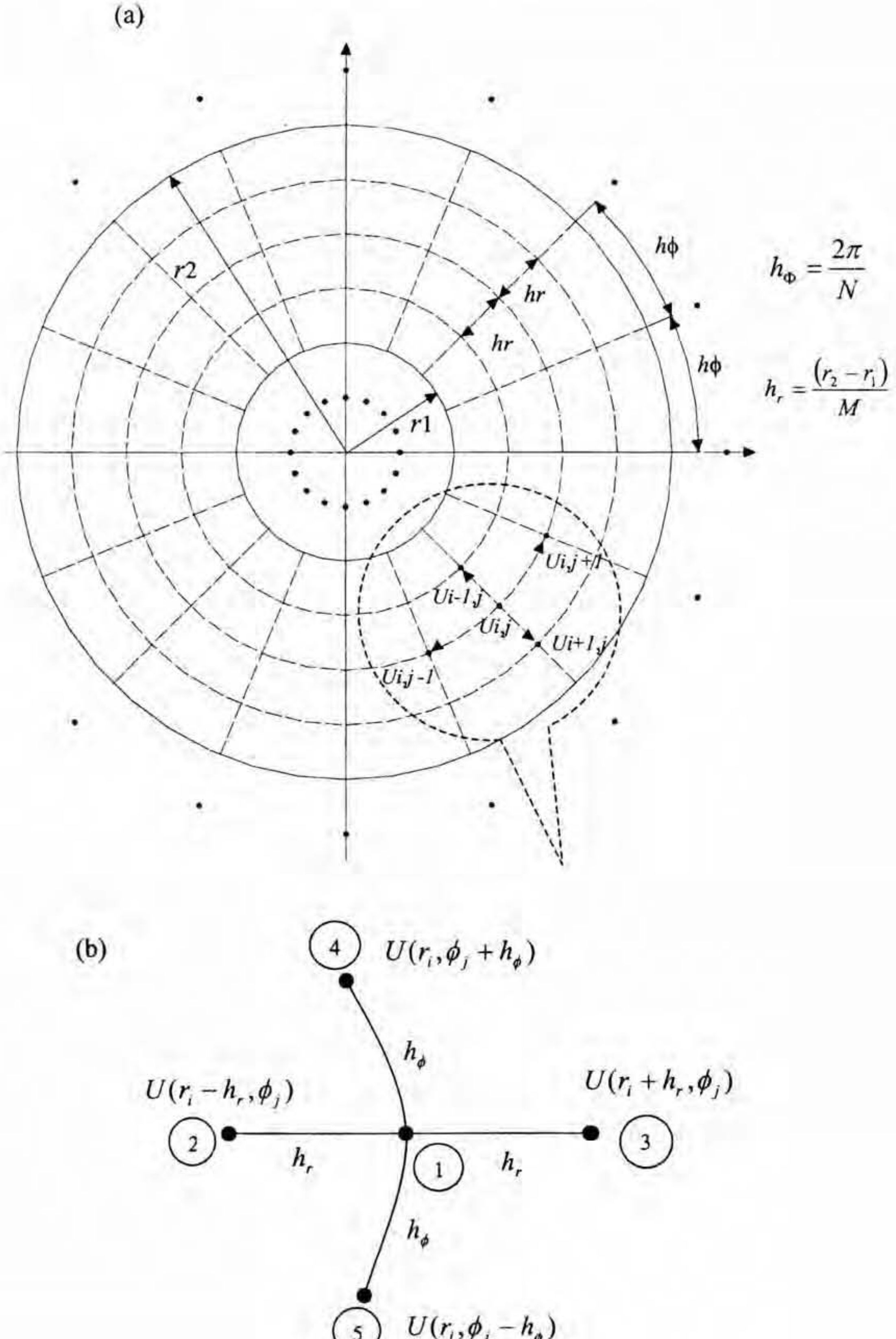

Fig. 1—(a) Grid configuration for application of finite difference method in coaxial waveguide (b) Extended view of nodes

If M and N are the number of nodes in the r and Φ directions, respectively, then the separations between the nodes in the two directions [from Fig.1 (a)] are

$$h_r = \frac{r_2 - r_1}{M}, h_\Phi = \frac{2\pi}{N}$$

In order to derive finite difference formulas for the differential term in Eq. (1), let us consider Taylor series expansion of function $U(r)$ about a point r_0 in forward (i.e. positive r) and backward (i.e. negative r) directions. Thus we have

$$U(r_0+\Delta r) = U(r_0) + \left.\frac{dU}{dr}\right|_0 \Delta r + \left.\frac{d^2U}{dr^2}\right|_0 \frac{(\Delta r)^2}{2!} + \left.\frac{d^3U}{dr^3}\right|_0 \frac{(\Delta r)^3}{3!} + \ldots\ldots\ldots \quad \ldots(2)$$

$$U(r_0-\Delta r) = U(r_0) - \left.\frac{dU}{dr}\right|_0 \Delta r + \left.\frac{d^2U}{dr^2}\right|_0 \frac{(\Delta r)^2}{2!} - \left.\frac{d^3U}{dr^3}\right|_0 \frac{(\Delta r)^3}{3!} + \ldots\ldots\ldots \quad \ldots(3)$$

Subtracting Eq. (2) from Eq. (3), the central difference approximation becomes

$$\left.\frac{dU}{dr}\right|_0 = \frac{U(r_0+\Delta r) - U(r_0-\Delta r)}{2\Delta r} - O(\Delta r)^2 \quad \ldots(4)$$

where

$$O(\Delta r)^2 = \frac{(\Delta r)^2}{6} U'''(r) + \frac{(\Delta r)^5}{120} U'''''(r) + \ldots\ldots\ldots \quad \ldots(5)$$

Neglecting higher order terms of $O(\Delta r)^2$, Eq. (4) becomes

$$\left.\frac{dU}{dr}\right|_0 = \frac{U(r_0+\Delta r) - U(r_0-\Delta r)}{2\Delta r} \quad \ldots(6)$$

Similarly, adding Eqs (2) and (3) and neglecting higher order terms, we get

$$\left.\frac{d^2U}{dr^2}\right|_0 = \frac{U(r_0+\Delta r) - 2U(r_0) + U(r_0-\Delta r)}{(\Delta r)^2} \quad \ldots(7)$$

The finite difference form of the different terms in Eq. (1) is obtained by using Eqs. (6) and (7) and Fig.1(b), and by replacing Δr with h_r, h_Φ for $\frac{\partial U}{\partial r}, \frac{\partial U}{\partial \Phi}$ and its second derivative terms as follows:

$$\frac{1}{r_i}\frac{\partial U}{\partial r} \approx \frac{U(r_i+h_r,\Phi_j) - U(r_i-h_r,\Phi_j)}{2r_i h_r} \quad \ldots(8)$$

$$\frac{1}{r_i^2}\frac{\partial^2 U}{\partial \Phi^2} \approx \frac{U(r_i,\Phi_j+h_\Phi) - 2U(r_i,\Phi_j) + U(r_i,\Phi_j-h_\Phi)}{r_i^2 h_\Phi^2} \quad \ldots(9)$$

$$\frac{\partial^2 U}{\partial r^2} \approx \frac{U(r_i+h_r,\Phi_j) - 2U(r_i,\Phi_j) + U(r_i-h_r,\Phi_j)}{h_r^2} \quad \ldots(10)$$

On substituting these values in the Helmholtz Eq. (1), the following difference equation is obtained

$$-\alpha_i U(r_i+h_r,\Phi_j) - \beta_i U(r_i-h_r,\Phi_j) - \gamma_i U(r_i,\Phi_j+h_\Phi) - \gamma_i U(r_i,\Phi_j-h_\Phi) + \left(\frac{2}{r_i^2 h_\Phi^2} + \frac{2}{h_r^2} - k^2\right) U(r_i,\Phi_j) = 0 \quad \ldots(11)$$

where

$$r_i = r_1 + (i-1)\frac{r_2 - r_1}{M}, \Phi_j = -\pi + (j-1)\frac{2\pi}{N} \quad \ldots(12)$$

and

$$\alpha_i, \beta_i = \frac{1}{h_r^2} \pm \frac{1}{2r_i h_r}, \gamma_i = \frac{1}{r_i^2 h_\Phi^2} \quad \ldots(13)$$

Equation (11) gives a set of simultaneous equations as i and j assumes values over the range $1 < i < M$ and $1 < j < N$. This set of simultaneous equations leads to a matrix equation of the form

$$([A] - k^2[I])[U] = [0] \quad \ldots(14)$$

where $[A]$ is a square matrix, $[I]$ is an identity matrix, $[U]$ is a column matrix, $[0]$ is a null matrix and k indicates the cut-off wavenumbers. Order of these matrices is $(M-p)\ (N-p)$, where p assumes values 2 (for TM modes) and 0 (for TE modes) depending on the boundary conditions. The eigenvalues are found from the solution of the characteristic equation

Table 1—Cut-off wavelengths of TM modes for $r_2=3r_1$

TM modes	λ (Evaluated by root finding)	λ (Finite difference)
TM_{01}	4.0577	4.0599
TM_{11}	3.8415	3.8435
TM_{21}	3.3636	3.3663

Table 2—Cut-off wavelengths of TE modes for $r_2=3r_1$

TE modes	λ (Evaluated by root finding)	λ (Finite difference)
TE_{11}	12.2331	13.4324
TE_{21}	6.4279	.3613
TE_{31}	4.5267	4.4663

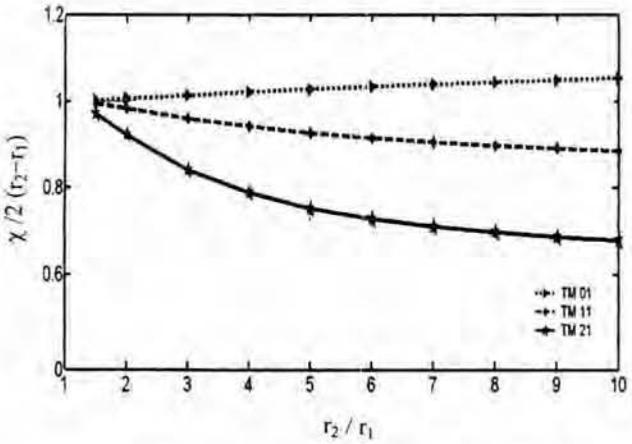

Fig. 2—Plots of normalized wavelengths versus outer-to-inner radii for TM modes

$$\det([A]-\chi[I])=0 \quad \ldots(15)$$

The boundary conditions satisfied by the different modes at $r=r_1$ and $r=r_2$ are

$U=0$ (Dirichlet condition for TM modes)

and

$\frac{\partial U}{\partial n}=0$ (Neumann condition for TE modes)

3 Numerical results

On comparing with the results by finding the roots from closed form expressions[1], for $r_2=3r_1$ an excellent match is found in the present study for cut-off wavelengths of TM modes and a relatively good match for TE modes, which are presented in Tables 1 and 2, respectively. Further, plots of normalized wavelengths versus the outer-to-inner radii r_2/r_1 are shown in Fig. 2 for the first 3 TM modes, which are in excellent agreement with the one depicted by Beaubien and Wexler[2]. In addition, similar plots for TE modes are shown in Fig. 3.

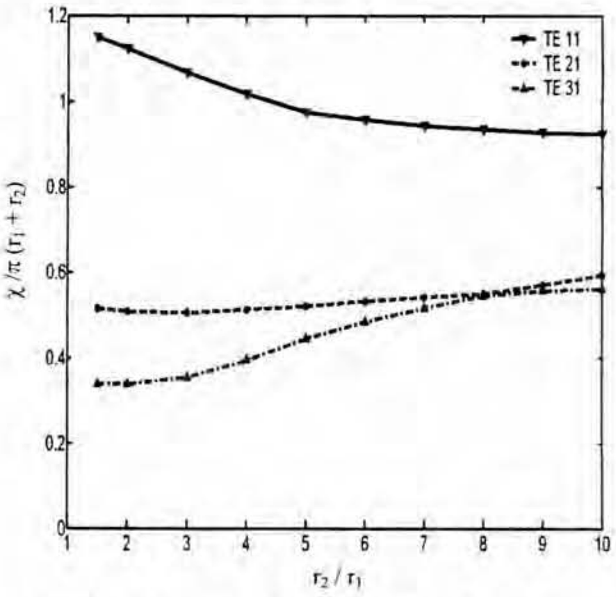

Fig. 3—Plots of normalized wavelengths versus outer-to-inner radii for TE modes

4 Conclusion

Finite difference technique is used efficiently to evaluate cut-off wave numbers of circular coaxial waveguides. The curvilinear rectangular grid formation for the selection of node points and subsequent application of Helmholtz equation in the polar form has shown excellent results. This technique of finite difference with curvilinear subsection may be used for the structures with curved and circular boundaries.

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