Non-Hertzian Approach to Predict Pressure Distribution in Gear Tooth Contact Problem

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Limitations of Hertz theory of contact to predict the pressure distribution in involute spur gear contact problem have been established by conducting 3-D photoelastic experiments. An approximate simplified non-Hertzian approach has been applied to predict the pressure distribution in such contact problems in which motion is a combination of rolling and sliding beyond pitch point and has been found to be satisfactory.

Introduction

Spur gears with involute profiles are designed to make nominal line contact under normal load. In action under the combined cyclic normal and tangential loading the line becomes a band due to elastic deformation of materials. When the gear tooth surfaces have crowning or mismatch due to different types of surface errors, the area of contact under stabilized working conditions becomes irregular in shape, having approximate axial symmetry in transverse direction only. Again, in gear tooth contact, rolling contact occurs at the pitch point and possibly near it, elsewhere the contact is that of sliding variety. The sliding or 'slip' velocity at any point between a pair of gear teeth is equal to the product of angular velocity of rolling (i.e., \( \omega_1 \), \( \omega_2 \)) and the distance of the point of contact from the pitch point and its direction changes from the arc of approach to the arc of recess. Also the point of contact between them itself moves in space, while at that point the two surfaces move relative to each other with a motion which combines both rolling and sliding. In such line contact problems the profiles are non-conforming in the plane of cross-section, but they do conform along the line of contact in the plane containing two 'equivalent cylindrical discs' whose radii continuously change with change in position of contact. The motion of rolling and sliding at a given instant in the meshing cycle can be reproduced by two circular discs of radii \( R_1, R_2 \), as shown in Figure 1, rotating with angular velocities \( -\omega_1 \), and \( +\omega_1 \), about fixed centers at \( C_1 \) and \( C_2 \). This, in turn, is equivalent to a cylinder rolling in contact with a semi-infinite elastic half-space.

In gear contact, there is always a tangential force responsible to transmit power and motion from the driver to the follower. According to Carter-Poritsky theory the slipping contact beyond pitch point introduces tangential frictional traction of varying magnitude opposite to the direction of motion. Invariably, for a pair of gears the materials are not same while designing them from bending strength consideration. This dissimilarity of materials introduces dissimilar tangential displacements, thereby introducing further slip in the contact zone. This 'slip' due to dissimilarity of materials of bodies in contact brings in further frictional tangential traction proportional to a constant \( K_{slip} \), and given by:

\[
K_{slip} = \frac{(1-2\nu_1)(1+\nu_2)E_2 - (1-2\nu_2)(1+\nu_1)E_1}{(1-\nu_1)E_2 + (1-\nu_2)E_1}
\]

where \( E \) and \( \nu \) are Young’s modulus and Poisson’s ratio, respectively of materials of two bodies in contact. Also the actual shape of contact area continuously changes with number of cycles of loading due to wear and plastic flow of materials until they attain stabilized condition, popularly known as 'elastic shakedown' of materials. Therefore, normal load combined with tangential traction and dynamic behaviour of gears in mesh affect the actual contact area and stresses under stabilized con-
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which has been adopted here for experimental investigation. For the second part of the problem the approximate simplified 2-D approach of Nayak\textsuperscript{8} has been applied to predict pressure distribution in spur gear tooth contact.

**Experimental Study by Photoelastic Method**

Two simple experiments of point contact using frozen-stress photoelastic approach have been conducted using: (a) A castolite sphere on a castolite plane; and (b) A steel sphere on a castolite plane. These two models are chosen to know the influence of dissimilarity of materials on deformation and internal stress distribution. The elastic and optical constants of castolite frozen models at room temperature are: modulus of elasticity, $E = 279.93$ N/mm$^2$, Poisson's ratio, $\nu = 0.50$; and material fringe value, in terms of stress, $f_s = 112.98$ N mm / fringe for green light source with $l = 5461$ Å. Steel with $E = 205$ Gpa, and $\nu = 0.30$ has been assumed to be incompressible when compared with deformation of castolite.

The specimens have been carefully chosen to simulate the condition of loading on elastic half-space. The final dimensions of models in which stresses are frozen photoelastically are: (a) A 19.05 mm diam castolite ball at the centre of a castolite block of 91.45 mm diam and 46.36 mm in depth; and (b) A 19.05 mm diam steel ball at the centre of a castolite block of 95.25 mm diam and 25.4 mm in depth. The load on both the models has been kept same as 174 N. After stresses have been frozen, central slices of 2.54 mm thick from each of the castolite blocks are used for photoelastic study in a Lens-Field Transmission Light Polariscope using white light source with green filter. The salient values of deformation and peak values of contact stresses are given in Table 1.

**Limitations of Hertz theory**

From experimentally obtained deformation contours for the models(Figure 2) it is observed that the contact planform is circular in nature, as predicted by Hertz theory. The deformation of the castolite block of second model is more localized in comparison to that of the first model, but the experimental values of deformation characteristics are in close agreement with the theoretical ones with some variation in maximum deformation for the castolite block of first model. This is expected, as the castolite sphere gets flattened creating more area of contact under the same load but its deformation at the centres becomes comparatively less as its size is in the ratio 1:5 when compared with the castolite block.
Table 1 — Salient values of deformation and peak values of contact stresses

<table>
<thead>
<tr>
<th>SI No.</th>
<th>Model</th>
<th>Parameters</th>
<th>Deformation characteristics</th>
<th>Maximum contact stress components</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Expt</td>
<td>Theo</td>
<td>per cent var</td>
</tr>
<tr>
<td>1</td>
<td>Castolite on castolite</td>
<td>$W_{\text{max}}$</td>
<td>0.80</td>
<td>0.862</td>
<td>25.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$a$</td>
<td>4.064</td>
<td>4.054</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Steel on castolite</td>
<td>$W_{\text{max}}$</td>
<td>1.181</td>
<td>0.87</td>
<td>8.6</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$a$</td>
<td>3.226</td>
<td>3.226</td>
<td></td>
</tr>
</tbody>
</table>

Theoretically, maximum deformation for both the bodies has to be same, though it is not in reality. Under concentrated contact, total deformation of the system remaining the same the larger body deforms more because of the effect of the penetration of the sphere into the plane surface. Also the experimental peak pressure for the same model is about 28.5 per cent more than that of the Hertz value and hence more deformation for the castolite block at the centre. For the second model the experimental and theoretical values of maximum deformation and area of contact are practically same. Figure 3 shows the experimentally obtained maximum shear stress distribution for both the models along the load line with Hertz values for comparison. The experimental values of maximum shear stress for the first model is about 19 per cent higher at a depth of 2 mm from the deformed surface and for the second model about 36 per cent higher at a depth of about 1.6 mm from the deformed surface than the corresponding theoretical values. This variation, in general, is mainly due to large deformation of finite castolite blocks and the discrepancy in variation in two experimental values is attributed to dissimilarity of materials in one of the models. The large difference in elastic constants of two bodies in contact in the second model induces plastic flow and ‘slip’ in the body of castolite block thereby inducing higher stresses (Table 1).

In view of these findings, it is felt that Hertz theory is not suitable for involute gear tooth contact, which is subjected to the combined normal and sliding force in addition to ‘slip’ due to dissimilarity of materials. Hence, non-Hertzian approach is more suitable to predict the pressure distribution in gear contact problems.

Prediction of Pressure by Non-Hertzian Approach

An alternative method using non-Hertzian approach has been adopted to predict the pressure distribution in gear tooth contact more accurately. The simplified non-
Hertzian approach, developed by Nayak\(^*\), based on 2-D Hertz theory has been applied to this problem and only relevant equations are given. The analysis and accuracy of the method are based on two major assumptions: (i) With contact area being reasonably narrow and symmetrical about the longitudinal axes the load per unit length \(N\), at any position \(x\) from the centre of contact is proportional to the product of the square of the half-width of contact \(b\) and equivalent curvature \(1/R\) at that position and (ii) The square of the half-width of contact is represented by a polynomial \(f(x)\) containing higher degrees of \(x\) such that the exponents of \(x\) are positive even integers only and a term free of \(x\) to represent the contact width at the centre. Accordingly:

\[
N_{x} = \frac{K b}{R_{y}}, \quad \text{where} \quad K = \frac{P}{\int \frac{1}{(b^2/R_y)} dx} \quad (2)
\]

where \(P\) represents the total normal load and \(a\) is the half-length of contact. The axial pressure at that position is given by:

\[
p_{x,0} = \frac{b_{x}}{R_{y}} \sqrt{\frac{E_{y} K}{\pi}} \quad (3)
\]

The equivalent radius of curvature \(R_x\) in the transverse direction and the equivalent Young’s modulus of contacting bodies \(E^*\) are given by:

\[
\frac{1}{R_x} = \frac{1}{R_{x1}} + \frac{1}{R_{x2}}, \quad \frac{1}{E^*} = \frac{1}{E_{1}} + \frac{1}{E_{2}} \quad (4)
\]

where \(R_{x1}\) and \(R_{x2}\) are radii of curvature in the transverse direction; \(E_1\) and \(E_2\) are Young’s modulus and \(E_{1}\) and \(E_{2}\) are Poisson’s ratios of two bodies in contact. The four-point polynomial, \(f(x)\), to represent the square of half-width of contact is given as:

\[
f(x) = (b^2 x^4 + C x^4 + D x^6) \quad (5)
\]

where \(A, B, C\) and \(D\) are constants to be determined in particular cases of contact. These constants are to be determined from experimentally obtained planforms, by choosing any three half-widths and the half-width at the centre and substituting them in Eq. (5).

**Application to Involute Gear Tooth Contact**

Based on above facts, it has been assumed that the planform of involute spur gear contact under stabilized operating conditions is non-Hertzian and is in the form of a band having arbitrariness in the rolling direction but having axial symmetry about transverse direction. As a first approximation, the gear tooth contact has been assumed to be equivalent to a radiused cylindrical roller in contact with a semi-infinite elastic flat surface. The shape of the contact area has been obtained from the
footprint made on a very thin plate of copper coating provided on the flat surface of a steel roller end by the cylindrical surface of another roller. The copper coating has been applied by using Steads Reagent solution, prepared by mixing in proportion of 10 g of cupric chloride, 40 g of magnesium chloride, and 20 cc of concentrated hydrochloric acid which was further diluted by adding 11 of alcohol. It has not been possible to get the actual contact planform for a pair of spur gear in contact under stabilized condition. However, a second approximation has been made for involute gear-contact problem by deviating from the ideal line contact condition in which assumed area of contact is about 10 to 15 per cent more than the ideal contact area. It may be noted that the ideal Lundberg profile has to be adopted for the surfaces to get a perfect rectangular contact area in order to obtain a uniform distribution of pressure along the entire length of the contact, which is practically not possible for gear tooth surfaces.

(i) Cylindrical Roller with Radiused Ends

The footprint shape has been obtained for a cylindrical roller of 19.05 mm radius in contact with the flat surface of another steel roller having edge radius of 1.5 mm under a load of 66.75 KN. Choosing four points including the centre point along the length of contact for values of $x$ and $y$, the arbitrary constants in Eq. (5) are found to be: $A = 0.648$, $B = -8.6 \times 10^{-7}$, $C = 6.4 \times 10^{-6}$, and $D = -5.0 \times 10^{-5}$, and hence Eq. (5) becomes:

$$y = 0.648 - 8.6 \times 10^{-7} x^2 + 6.4 \times 10^{-6} x^3 - 5.0 \times 10^{-5} x^4; \ldots (6)$$

With $R = 19.05$ mm, $a = \pm 8.15$ mm, $E'' = 113.2$ GPa for steel and $P = 66.75$ KN, the value of $K$ is equal to 114162.38 N/mm$^2$ and the axial pressure distribution, Eq. (3) becomes:

$$P_{xy} = 3366.78 b; \ldots (7)$$

The experimental semi-width of contact along with the predicted values from Eq. (6) as well as the predicted axial pressure distribution from Eq. (7), normalized with respect to corresponding values at the centre are shown in Figure 4 (a) and (b). The predicted half-width of contact practically matches with the experimental results. The predicted pressures at the centre and at the end point are $2.71 \times 10^6$ N/m$^2$ and $3.12 \times 10^6$ N/m$^2$, respectively, giving a stress concentration of about 1.15 at the end, which seems reasonable, whereas Hertz theory predicts a uniform pressure of $2.78 \times 10^6$ N/m$^2$ throughout the contact length for the same load. In comparison with gear tooth contact the maximum pressure may not occur at the end point, but somewhere within the contact zone under smooth operating condition. Still, this simulation gives an idea about the pressure variation likely to occur in gear contact problems.

(ii) Involute Spur Gear Tooth Contact

A pair of 20\' full depth involute spur gears of 2.5 mm module having 180 mm pitch circle diam and 72 numbers of teeth for the gear and 90 mm pitch circle diam and 36 numbers of teeth for the pinion in mesh
with pinion rotating at 1200 rpm, as shown in Figure 1, is considered. The material for both pinion and gear has been assumed to be of plain carbon steel with 205.5 GPa and 0.3 as Young’s modulus and Poisson’s ratio, respectively. The face width of both the wheels is taken to be 20 mm. The wheels rotate with angular velocities \( \omega_1 = 125.6 \text{ rad/s} \) and \( \omega_2 = 62.8 \text{ rad/s} \) about their respective axes. The path of contact is a straight line and is found to be equal to 12.94 mm with standard specifications for full depth involute teeth. When the contact at any instant is at a point O away from the pitch point the motion is the combined effect of rolling and sliding with sliding velocity equal to 4.22 m/s. As mentioned earlier, this motion at the point O in the meshing cycle can be reproduced by two circular discs of same material as the pair of gears of radii \( I_1 = 37.5 \text{ mm} \) and \( I_2 = 26.25 \text{ mm} \) and thickness 20 mm each rotating with angular velocities \(-125.6 \text{ rad/s}\) and \(62.8 \text{ rad/s}\) about fixed centres at \( I_1 \) and \( I_2 \) (Figure 1). For the contact at O at a particular instant for this pair of gears it is statically equivalent to a cylindrical disc of equivalent radius \( R = 15.44 \text{ mm} \) [Eq. (4)], in contact with elastic flat surface of the same material with ideal contact length of 12.94 mm, which is assumed to be equal to the path of contact for the line contact problem. Under an assumed load of 21.8 KN Hertzian width of contact becomes 0.541 mm with an aspect ratio \( a/b = 11.96 \) for equivalent Young’s modulus, \( E' = 113.2 \text{ GPa} \), and the corresponding Hertzian peak pressure is 1.984 GPa.

In action the line becomes a band due to elastic deformation of materials under stabilized condition. This band may become an irregular figure or approach elliptical shape when surfaces have mismatch or a bit of crowning and the exact shape of this unknown planform depends upon the initial accuracy of surface as well as wear and deformation of materials during the period of shake down. In this process, actual area of contact may increase by 10-15 per cent with deviation in width and length of contact from ideal line contact problem. To take care of such possibilities we have taken two cases: (i) The contact area deviates from rectangular shape with width at the centre remaining practically same, i.e., \( b = 0.55 \text{ mm} \) and half length increasing from 6.47 to 7.5 mm, i.e., \( a = 7.5 \text{ mm} \), \( a/b = 11.96 \); (ii) The shape of the contact planform deviates from elliptical shape with \( b = 0.675 \text{ mm} \), \( a = 7.5 \text{ mm} \) and \( a/b = 11.11 \) (Figure 5). In both the cases the aspect ratio has been maintained more or less same as that of ideal rectangular contact for comparative study, but otherwise arbitrary and total area has been increased by about 12 per cent from the ideal value. Using the approach adopted for radiused roller in contact the corresponding values of constants for the case of (a) are: \( A = 0.3025 \), \( B = 2.525 \times 10^{-3} \), \( C = 1.099 \times 10^{-4} \), \( D = -4.73 \times 10^{-6} \) and \( K = 75114.94 \), from which equations for \( b \), \( \rho_{5y} \) take the form:

\[
(b)^2 = 0.3025 + 2.525 \times 10^{-3} \rho_{5y}^2 + 1.099 \times 10^{-4} \rho_{5y}^4 - 4.73 \times 10^{-6} \rho_{5y}^6 \tag{8}
\]
For the second case (b) in which there is deviation from elliptical contact area, the constants are: 
\[ A = 0.455625, \quad B = -1.6 \times 10^{-6}, \quad C = 4.89 \times 10^{-4}, \quad D = -6.65 \times 10^{-4} \text{ and } K = 75879.93 \]
from which equations for \( b_x \) and \( p_{x,y} \) become:

\[
(b_x)^2 = 0.455625 - 1.6 \times 10^{-6} x^2 + 4.89 \times 10^{-4} x^4 - 6.65 \times 10^{-4} x^6 \quad (10)
\]

\[
P_{x,y} = 3366.78 \quad (11)
\]

The predicted half-width of contact from Eqs. (8) and (10)] and axial pressure distribution from Eqs. (9) and (11)] along the contact length for both the cases have been shown in Figure 5(a) and (b). It is observed that for the first case the maximum pressure at the centre is about 1.85 GPa, whereas it is about 2.02 GPa at \( x = 5.0 \text{ mm} \) which is about 9.2 per cent higher than the pressure at the centre. For the second case, maximum pressure of 2.28 GPa occurs at the centre giving a stress concentration of about 1.15, which seems to be reasonable.

Conclusions

The work reported is in two parts. In the first part, limitations of Hertz theory of contact has been discussed using the 3-D photoelastic technique. From two simple experiments under point contact, it has been observed that there is large variation of actual contact pressure when compared with that of the Hertz, though deformation characteristics are in close agreement with theoretical prediction. From these experiments, it has been concluded that Hertz theory may not be suitable to predict pressure distribution in gear tooth contact in which there is combined motion of rolling and sliding beyond the pitch point. In addition the dissimilarity of materials of two wheels introduces further slip in the contact zone. Hence the problem has been treated as a non-Hertzian contact problem. In the second part, an approximate simplified non-Hertzian approach has been applied to predict pressure distribution in involute spur gear contact problem.

As it has not been possible to get the actual area of contact under stabilized conditions the area of contact beyond pitch point for a pair of spur gears has been assumed to deviate from the ideal rectangular contact by 10-15 per cent to predict the pressure distribution. Taking two different assumed shapes, as shown in Figure 5(a), axial pressure distributions have been predicted[Figure 5(b)]. From the analysis, it has been shown that pressure distribution does not remain uniform throughout and there is some stress concentration of about 1.10 to 1.15 per cent at some points under normal loading. This distribution is likely to undergo further change when the slip at the contact zone due to tangential traction and dissimilarity of materials are included in the analysis. Because of such complexities of the problem it has been suggested that this non-Hertzian approach is better suited to predict the gear tooth failure due to pitting fatigue. However, further investigations are required to include the influence of slip due to tangential traction and dissimilarity of materials to accurately predict the pitting fatigue life of gears.

References

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