Stable block Toeplitz matrix for the processing of multichannel seismic data

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Computation of deconvolution operators in the case of single channel sectioned input/multichannel seismic data involves the inversion of a block Toeplitz matrix. The inversion of such a matrix poses several problems. It is well established that the error energy which measures the well posedness of the matrix is seen to decrease with an increase in the filter length. However, with an increase in filter length the condition number of the associated matrix increases. This means that there is a trade off between ill posedness and accuracy. The ill-posed problem has been made well posed by a process of (1) normalization of the block Toeplitz matrix and (2) by adding prewhitening parameter. The prewhitening parameter is taken as a few per cent of arithmetic or the geometric mean of the main diagonal of the block Toeplitz matrix. Application to a synthetic as well as field seismic data shows that the condition number of the associated block Toeplitz matrix is reduced by a process of normalization and adding prewhitening parameter. Further it is observed that the condition number is smaller when the prewhitening parameter is taken as a few per centage of geometric mean as compared to the arithmetic mean. Stabilizing the matrix following the above procedure will help in obtaining stable as well as accurate deconvolution operators.

[Key words: Toeplitz matrix, multichannel seismic, deconvolution, filter length]

Introduction

Multichannel data are acquired in many fields e.g. reflection seismic, seismology, astronomy, sonar and radiometry. Various multichannel filters have been designed for the suppression of coherent noise, stacking problems and velocity analysis. One algorithm used for the processing of multichannel data is known as the LWR (Levinson-Wiener-Robinson) algorithm which is fast and calculates the required prediction filter for a time-invariant sequence. However, for the time-varying input sequence alternative methods to the LWR algorithm such as the gradient methods have been proposed. The design of least squares deconvolution operators for multichannel seismic data involves the inversion of the block Toeplitz matrix. The computation of the combined deconvolution operators for the time-varying sequence being sectioned into piece wise stationary sections also involves the block Toeplitz matrix. This block Toeplitz matrix is structurally similar to the single channel autocorrelation matrix but with a difference that its elements are matrices rather than scalars and the main property is that all the sub matrices are identical. In this investigation, a detailed study on the general properties of the block Toeplitz matrix regarding the stability of the solution and accuracy of the estimated parameters, through the analysis of the condition number has been carried out. It has been well established that when the condition number is nearly one, the system is said to be well conditioned i.e. stable. However, unstable systems are associated with large condition number and are termed ill conditioned. Hence, it becomes necessary to investigate the behaviour of condition number of a matrix with respect to various factors such as normalization of the block Toeplitz matrix and adding prewhitening parameter. The stability of the block Toeplitz matrix has been discussed in light of the prewhitening parameter.

Materials and Methods

The computation of the deconvolution operators for a single channel sectioned input and multi channel data are designed using normal equations that show up the property of the block Toeplitz matrix. The block Toeplitz matrix is mathematically expressed as

\[
R = \begin{bmatrix}
P(0) & P(1) \\
P(-1) & P(0)
\end{bmatrix}
\]  

\[\ldots (1)\]
where \( P(0) = \begin{bmatrix} R_{ii}(0) & R_{ij}(0) \\ R_{ji}(0) & R_{jj}(0) \end{bmatrix} \)

and

\[
P(1) = \begin{bmatrix} R_{ii}(1) & R_{ij}(1) \\ R_{ji}(1) & R_{jj}(1) \end{bmatrix}
\]

The elements of the matrix are correlation functions where \( i/j \) indicate the number of channels or sections. Suppose we have two sections/channels say \( X_1(t) \) and \( X_2(t) \), then the correlation function is expressed as:

- \( R_{11}(\tau) = E[X_1(t+\tau)X_1(t)] \) auto-correlation function of \( X_1(t) \)
- \( R_{12}(\tau) = E[X_1(t+\tau)X_2(t)] \) cross-correlation function of \( X_1(t) \) and \( X_2(t) \)
- \( R_{21}(\tau) = E[X_2(t+\tau)X_1(t)] \) cross-correlation function of \( X_2(t) \) and \( X_1(t) \)
- \( R_{22}(\tau) = E[X_2(t+\tau)X_2(t)] \) auto-correlation function of \( X_2(t) \)

where \( \tau = \) lag, i.e. \( \tau = 0, 1, 2, 3, \ldots \) When \( \tau = 0 \) and 1 we get a two-length filter which has a block-Toeplitz matrix represented by Eq. (1) and each element of the matrix \( R \) is a sub matrix. In the design of accurate and stable deconvolution operators mathematical operations need to be performed on the associated block Toeplitz matrix, \( R \). The approach taken in this paper is to quantify the stability of the block Toeplitz matrix through the analysis of its condition number. The condition number, \( CN \), measures the well posedness of the system and is the ratio of the maximum eigen value to the minimum eigen value:

\[
CN = \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} \quad \ldots (2)
\]

The system is said to be well conditioned if the condition number is nearly one. However, in most seismic deconvolution studies the block Toeplitz matrix associated with its design are associated with large condition number. Hence, the behaviour of the condition number is investigated with respect to (1) normalization of the block Toeplitz matrix, (2) prewhitening parameter, and (3) length of the filter.

**Normalization of the block Toeplitz matrix**

In this study the estimated covariance matrix i.e. the block Toeplitz matrix is normalized\(^5\), where the maximum likelihood method was applied to several sonic data and the covariance matrix obtained was normalized by:

\[
\hat{R}_{ij}(\tau) = \left( \prod_{j=1}^{m} R_{ij}(\tau) \right)^{1/m} \left( R_{ij}(\tau)/R_{ii}(\tau)R_{jj}(\tau) \right) \quad \ldots (3)
\]

where \( R_{ii}(\tau) \) is the autocorrelation of the \( i^{th} \) channel and \( R_{ij}(\tau) \) is the cross correlation between \( i^{th} \) channel and \( j^{th} \) channel and \( \left( \prod_{j=1}^{m} R_{ij}(\tau) \right)^{1/m} \) is the geometric mean of the diagonal elements of the block Toeplitz matrix.

This normalization procedure forces the estimated covariance matrix to have equal diagonal elements, a property of the Toeplitz matrix.

**The prewhitening parameter**

The prewhitening parameter is applied to avoid the singularity arising from the zero eigen value. In other words, we add a small amount of white noise to stabilize the covariance matrix i.e. prewhitening parameter is added to the main diagonal to make the inverse scheme stable. Prewhitening the Wiener filter improves the output signal to ambient noise ratio, but at the same time it reduces the resolution\(^7\). The selection of appropriate prewhitening parameter is a very tedious procedure as seen in most methods like the ridge regression method\(^8\), the singular value decomposition\(^9,10\) and the maximum likelihood method\(^11\). In routine seismic processing the prewhitening parameter is chosen as a few per cent of the arithmetic mean of the main diagonal. As the geometric mean is known to restore the overall power of the covariance matrix, the prewhitening parameter is taken as a few per cent of the geometric mean of the main diagonal. Adding prewhitening parameter to the normalized block Toeplitz matrix results in

\[
\hat{R}_{ij}(\tau) = R_{ij}(\tau) + \theta I \quad \ldots (4)
\]

where \( \theta \) is prewhitening parameter and \( R_{ij}(\tau) \) is normalized block Toeplitz matrix given by Eq. (3) and \( I \) is the identity matrix.

**Length of the filter**

The stability of the block Toeplitz matrix associated with different filter lengths has been studied by analyzing its condition number. A well-
established fact is that the error energy associated with the design of optimum filter decreases with an increase in the filter length. In other words, we can expect the solution of the problem to be well posed by increasing the length of the filter. However, its accuracy needs to be tested and we shall demonstrate it through a numerical example.

**Application to Data and Results**

**Field data**

A six-channel unprocessed marine seismic data collected from the western continental margin of India has been taken for the study (Fig. 1). The matrix of auto and cross correlation associated with the design of deconvolution operators for the two input traces has been obtained. The condition number for the above block Toeplitz matrix has been computed first by normalization of the matrix using Eq. (3). The condition number has been obtained for both the cases i.e. with normalization and without normalization and it is observed that the condition number has reduced by a process of normalization (Figs 2 and 3). The effect of prewhitening parameter taken as a few per cent of arithmetic or geometric mean on the condition number has also been dealt with. The condition number has been obtained for the normalized block Toeplitz matrix adding prewhitening parameter (Eq. 4). The prewhitening parameter has been taken as 1 to 20% of the arithmetic or geometric mean of the main diagonal of the matrix and Figs 2 and 3 show the plot of condition number versus prewhitening parameter for the two cases. It is further observed that with an increase of prewhitening parameter the condition number reduces in both cases. However, reduction in condition number is more in the case of geometric mean as compared to that of the arithmetic mean.

Finally, the effect of the filter length i.e. the size of the block Toeplitz matrix on the condition number is analyzed. The condition number for different filter lengths has been computed (Table 1), which shows

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**Fig. 1**—Six channels unprocessed marine seismic traces

**Fig. 2**—Condition number vs prewhitening parameter in the case of arithmetic mean for synthetic data (a) with normalization and (b) without normalization

**Fig. 3**—Condition number vs prewhitening parameter in the case of geometric mean for synthetic data (a) with normalization and (b) without normalization.
Table 1—Condition number for different filter lengths in case of marine seismic data

<table>
<thead>
<tr>
<th>Filter length</th>
<th>Condition number with prewhitening</th>
<th>Condition number without prewhitening</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>52</td>
<td>53</td>
</tr>
<tr>
<td>2</td>
<td>480</td>
<td>503</td>
</tr>
<tr>
<td>3</td>
<td>14753</td>
<td>302712</td>
</tr>
<tr>
<td>4</td>
<td>16758</td>
<td>1912396</td>
</tr>
</tbody>
</table>

that the condition number has increased with an increase in the filter length.

**Synthetic example**

A time series can be represented by the convolution of wavelets and noise series i.e. the Woldian decomposition theorem. Using this concept synthetic seismic trace is generated which is mathematically represented by

\[ X(t) = \alpha_i \ast A(t) + k \cdot \beta_i \ast B(t) \quad \ldots (5) \]

where \( \alpha_i \) is the source wavelet, \( \beta_i \) is the noise wavelet and \( k \) is the signal to noise ratio of the trace respectively. \( A(t) \) is the reflectivity series generated using random numbers between \([0,1]\) and \( B(t) \) is the random Gaussian noise series.

The choice of the wavelets depends on the nature of the trace that one would require. In several seismic explorations the Ricker wavelet is generally used\(^{18}\). Synthetic traces have also been generated using the minimum delay wavelet for the source function. Hence, in this study the synthetic seismic traces have been obtained for a given set of controlling parameters, i.e.,

\[ \alpha_i = 5.,4.,3.,-1.,0.5 \quad \beta_i = 4.,-2.,1.,-1.,0.5 \quad k = 1/1.5 \]

Figure 4 shows the plot of the trace that has been obtained and has been divided into two sections as reported earlier\(^5\). For these two sections the computation of the filter coefficients requires the block Toeplitz matrix\(^4\). The procedure\(^4\) as discussed has been followed and for different lengths of the filter the error energy or the filter performance has been computed\(^3\). After normalizing the block Toeplitz matrix and adding some prewhitening the condition number has been obtained. The results are plotted in Fig. 5. From the figure it is clear that as the length of the filter increases, the error energy decreases but the condition number is increasing. We see that for a filter length of 11 the two curves are cutting each other. The length of the filter for which the error energy becomes zero has been given earlier\(^4\). For this example, if we were to look for accurate filters we would need to compute a filter of length 19. However, since we need to make a compromise the intersection point could be assumed to be an optimum length that takes care of the stability and accuracy of the filter design.

**Discussion**

In this study we have considered a six-channel marine seismic data and synthetic seismic traces and for achieving the stability of the deconvolution operators. Single channel sectioned input and multichannel seismic deconvolution involve the inversion of the block Toeplitz matrix. To obtain a stable solution, the block Toeplitz matrix associated with the deconvolution operators has been normalized.
to get low values of the condition number as this measures the well posedness of the problem. Further, the stability of the algorithm has been achieved by adding the prewhitening parameter. It is well known that the prewhitening parameter prevents the inverse from blowing. However, it is observed that with an increase in the prewhitening parameter the condition number decreases and also it has been found that the prewhitening parameter computed from the geometric mean as compared to the arithmetic mean of the elements of the main diagonal of the block Toeplitz matrix reduces the condition number further. We have also seen that with an increase in the filter length the condition number also increases. However, it is well known that with an increase in filter length the error energy decreases thereby improving the performance of the filter\(^2\). A trade off between well posedness and accuracy of the problem exists and one needs to be careful in choosing the length of the filter in such a way that stability as well as accuracy is maintained.

This seismic data has been used to remove the multiples using the adaptive deconvolution\(^19\). The combined convergence factor for the adaptive algorithm is obtained. The auto and cross correlation matrix has been stabilized using the same procedure to obtain the maximum eigen value, that is used to obtain the convergent factor of the adaptive algorithm.

The procedure can be applied in various geophysical studies but care should be taken for appropriate compromise between the stability and accuracy of the estimated parameter. This study will be very useful in designing deconvolution operators for a single channel sectioned input/multi channel data that will account for its stability, accuracy and well posedness.

References