A modified quarter point element for fracture analysis of cracks

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A modification to the quarter-point crack tip element for the computation of stress intensity factors in cracked plates subjected to mixed mode loading is presented. The standard quarter-point singular element is modified such that the near tip crack opening displacement satisfies a known constraint; the coefficient of the term in William’s expression which is a linear function of the distance to the tip must vanish in order to have the singularity at the crack tip. Stress intensity factors computed using the modified quarter-point crack tip element in conjunction with the displacement correlation technique gives better results as compared to those obtained by the standard quarter-point element. The technique basically involves the same guidelines as that of the standard quarter-point element, where 6-noded quarter-point elements are adopted to mesh the singular region around the crack tip and 8-noded quadrilateral elements are adopted to mesh rest of the non-singular region. Based on the shape functions of the proposed modified quarter-point crack tip element the equations for computing the crack tip opening displacement and the stress intensity factors are presented taking into account the nodal displacement values of the elements around the crack tip. The results obtained using the modified quarter-point crack tip element are shown to be significantly more accurate than those obtained using the standard quarter-point element for both mode-I and mixed-mode problems.

IPC Code: G01N

In the numerical modeling of fracture, correct representation of the local stress and displacement fields in the crack tip region is essential for accurate evaluation of stress intensity factors (SIFs). For a cracked geometry, William’s\(^1,2\) solution for the displacement field in the neighborhood of the tip is

\[
\Delta u_k(r, \theta) = a_k + b_k(\theta) r^{1/2} + c_k(\theta) r + d_k(\theta) r^{3/2} + L \, k = 1, 2, \ldots (1)
\]

where \(r\) is the distance to the crack tip and \(\theta\) is the direction emanating from the tip as shown in Fig. 1. Thus, the crack opening displacement (COD) is

\[
\Delta u_k(r, \theta) = \Delta b_k(\theta) r^{1/2} + \Delta c_k(\theta) r + \Delta d_k(\theta) r^{3/2} + L \, k = 1, 2, \ldots (2)
\]

In the finite element modeling of discrete cracks, the standard approach is to incorporate the \(\sqrt{r}\) - singularity by means of the quarter-point\(^3,5\) (QP) element developed by Henshell and Shaw\(^6\) and Barosum\(^7\). Use of the QP singular elements has significantly improved the performance and over the years the element has been modified to enhance the performance. Recently, it has been proved by Gray and Paulino\(^8\) that irrespective of the problem geometry or boundary conditions, coefficients of the linear terms in William’s expansion\(^1,2\) are related by

\[
c_k(\pi) = c_k(-\pi), \ldots (3)
\]

for \(\Delta u_k\) on the crack faces. So the linear term vanishes from the expression of crack tip opening displacement evaluated along the crack faces,

\[
\Delta u_k(r) = u_k(r, \pi) - u_k(r, -\pi). \ldots (4)
\]

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The standard QP element fails to satisfy the constraint given in Eq. (3), and hence Eq. (4). Therefore by forcing the linear term in \( \Delta u_k \) to be zero, we expect to have a more accurate analysis at the crack tip region. Given the importance and interest in fracture mechanics of solids and structures, it is somewhat surprising that this simple analytical result went unobserved for so long.

**Modified Quarter Point Element**

The two-dimensional QP element shown in Fig. 2 is based upon the six noded quadratic element. For \( t \in [0,1] \) with the crack tip at \( t=0 \), the shape functions for this element along the crack edge (corresponding to nodes 1, 2 and 4) are given by

\[
\psi_1(t) = (1-t)(1-2t) \\
\psi_2(t) = t(2t-1) \\
\psi_4(t) = 4t(1-t) 
\]

As \( (\Delta u_1, \Delta u_2) = (0,0) \) at the crack tip (node 1), the representations of the crack edge, \( \Gamma(t) \) (corresponding to nodes 1, 2 and 4) and COD, \( \Delta u_k \) are

\[
\Gamma(t) = [(x,y) = x_1 \psi_1(t) + x_2 \psi_2(t) + x_4 \psi_4(t), y_1 \psi_1(t) + y_2 \psi_2(t) + y_4 \psi_4(t)] \\
\Delta u_1(t) = \Delta u_2^2 \psi_2(t) + \Delta u_4^4 \psi_4(t) \\
= -(\Delta u_2^2 - 4\Delta u_4^4)t + (2\Delta u_2^2 - 4\Delta u_4^4)t^2 \\
\Delta u_2(t) = \Delta u_3^2 \psi_2(t) + \Delta u_4^4 \psi_4(t) = -(\Delta u_3^2 - 4\Delta u_4^4)t \\
+ (2\Delta u_3^2 - 4\Delta u_4^4)t^2 
\]

where \( (x_1, y_1) \), \( (x_2, y_2) \), and \( (x_4, y_4) \) are the coordinates of the three nodes 1, 2 and 4 respectively (shown in Fig. 2) and \( \Delta u_k \) the nodal values of the COD at \( j \)-th node. As demonstrated earlier by moving the mid-node coordinates \( (x_i, y_i) \) three fourths of the way towards the tip, the parameter \( t \) becomes \( (r/L)^{1/2} \), with \( L \) being the distance from \( (x_1, y_1) \) to \( (x_2, y_2) \). As a consequence, the first order term in \( \Delta u_k \), which is \( t \), is the correct square root of distance \( (r/L)^{1/2} \). However the next term which is \( t^2 \), is \( r/L \). This term should vanish to give a more accurate representation of SIFs. So the new shape functions, \( \hat{\psi}_2, \hat{\psi}_4 \) as given in Eq. (7) below, are assumed for nodes 2 and 4 keeping the leading term \( t = \sqrt{r} \) intact and replacing the \( (r/L)^{1/2} \) term with \( (r/L)^{1/2} \).

\[
\hat{\psi}_2(t) = t(2t-1) + 2t(1-t)(1-2t)/3 = \frac{1}{3}(4t^3 - t) \\
\hat{\psi}_4(t) = 4t(1-t) - 4t(1-t)(1-2t)/3 = \frac{8}{3}(t^3 - t) 
\]

This simple modification to the shape functions for nodes 2 and 4 accomplishes the cancellation of \( t^2 \), without disturbing the interpolation, i.e.,

\[
\hat{\psi}_2(0) = 0, \quad \hat{\psi}_2(1/2) = 0, \quad \hat{\psi}_2(1) = 1 \\
\hat{\psi}_4(0) = 0, \quad \hat{\psi}_4(1/2) = 1, \quad \hat{\psi}_4(1) = 0 
\]

Also as is shown in Fig. (3), this alteration does not change the shape functions \( \psi_2(t) \) and \( \psi_4(t) \) used for computing COD in Eq. (6), radically.
Numerical Implementations

SIFs provided by the QP element (both modified and standard) will be calculated by means of the displacement correlation technique (DCT). However, the point to be noted here is to evaluate the quality of modified QP element by means of very simple technique such as DCT. The general expression of the mixed-mode SIFs, $K_I$ and $K_{II}$ by means of DCT are given by

$$K_I = \frac{G}{\kappa + 1} \lim_{r \to 0} \frac{2\pi}{r} \Delta u_2$$

$$K_{II} = \frac{G}{\kappa + 1} \lim_{r \to 0} \frac{2\pi}{r} \Delta u_1$$

where $\Delta u_2$ is the COD in the coordinate system associated with the crack tip under consideration, $G$ is the shear modulus, $\nu$ is the Poisson’s ratio, $\kappa = 3 - 4\nu$ and $\kappa = (3 - \nu)/(1 + \nu)$ under plane strain and plane stress conditions respectively. By using the modified QP shape functions in Eq. (7), we obtain

$$\Delta u_k = \Delta u_k^2 \psi_2(t) + \Delta u_k^4 \psi_4(t)$$

$$= \frac{4}{3} \left(\Delta u_k^2 - 2\Delta u_k^4\right) t + \frac{1}{3} \left(8\Delta u_k^4 - \Delta u_k^2\right)$$

Use of Eq. (10) in Eq. (9) and noting that $t = \sqrt{r/L}$ yields

$$K_I = \frac{G}{3(\kappa + 1)} \lim_{r \to 0} \frac{2\pi}{L} \left(8\Delta u_k^4 - \Delta u_k^2\right)$$

$$K_{II} = \frac{G}{3(\kappa + 1)} \lim_{r \to 0} \frac{2\pi}{r} \left(8\Delta u_k^4 - \Delta u_k^2\right)$$

Thus, SIFs are given directly in terms of the nodal values of the COD at the crack tip element. Since (Fig. 3) the interpolation characteristics of the proposed modified shape functions, $(\psi_2, \psi_4)$ are not radically different from the standard shape functions $(\psi_2, \psi_4)$, Eq. (11) can be used directly in conjunction with the nodal values obtained by adopting the standard QP element and can obtain more accurate predictions of the mixed-mode SIFs, $K_I$ and $K_{II}$. As shown in the numerical examples section below, the results obtained by Eq. (11) will be more accurate when compared with the Eq. (9), which is obtained based on the standard QP shape functions $(\psi_2, \psi_4)$.

Hence, the main advantage of the proposed technique is that no major modifications are required for its implementation in conjunction with any of the existing commercial software like (ANSYS, ABAQUS etc.) or research software like (FRANC2D/L etc.), except replacing Eq. (9) with Eq. (11) for evaluation of the mixed-mode SIFs, $K_I$ and $K_{II}$.

Numerical Examples and Discussion

In order to assess the various features of the modified QP element both mode-I and mixed-mode examples are presented. After obtaining the nodal displacement values of the crack tip elements the mixed-mode SIFs, $K_I$ and $K_{II}$ are evaluated using Eq. (11). Singular QP 6-noded elements are used in the region around the crack tip, and 8-noded serendipity elements are used for rest of the problem domain. In all the examples Young's modulus $E = 10 \times 10^6$ units, Poisson’s ratio $\nu = 0.25$ units are used, and plane stress conditions are assumed for the numerical analysis.

Example 1: Edge-Cracked Plate
The example involves an edge-cracked plate as shown in the Fig. 4a fixed at the bottom and subjected to far field shear stress $\tau^\infty = 1$ unit applied at the top. The
Table 1 — Comparison of SIFs for an edge-cracked plate

<table>
<thead>
<tr>
<th>Method</th>
<th>Predicted value</th>
<th>Error (%)</th>
<th>K_I</th>
<th>Predicted value</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MCCI</td>
<td>33.8451</td>
<td>0.6172</td>
<td>4.7721</td>
<td>6.7645</td>
<td></td>
</tr>
<tr>
<td>Standard QP element in conjunction with DCT</td>
<td>33.9156</td>
<td>0.4411</td>
<td>4.3140</td>
<td>3.8521</td>
<td></td>
</tr>
<tr>
<td>Modified QP element in conjunction with DCT</td>
<td>34.0721</td>
<td>0.2416</td>
<td>4.3765</td>
<td>2.3713</td>
<td></td>
</tr>
</tbody>
</table>

*K_I = 34.1123, K_II = 4.47 obtained by J–Integral method is taken as reference

Table 2 — Comparison of SIFs for a flat plate with central hole with the crack initiating from hole

<table>
<thead>
<tr>
<th>Method</th>
<th>Predicted value</th>
<th>Error (%)</th>
<th>K_I</th>
<th>Predicted value</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MCCI</td>
<td>15296.2178</td>
<td>0.7825</td>
<td>-2.4981</td>
<td>4.1701</td>
<td></td>
</tr>
<tr>
<td>Standard QP element in conjunction with DCT</td>
<td>15531.1321</td>
<td>0.7171</td>
<td>-4.3535</td>
<td>2.0903</td>
<td></td>
</tr>
<tr>
<td>Modified QP element in conjunction with DCT</td>
<td>15345.4155</td>
<td>0.4630</td>
<td>-4.2103</td>
<td>1.4309</td>
<td></td>
</tr>
</tbody>
</table>

*K_I = 15416.2171, K_II = -4.2691 obtained by J–Integral method is taken as reference

plate has a length \( L = 16 \) units and width \( W = 7 \) units and crack length \( a = 3.5 \) units. Finite element discretization adopted as shown in Fig. 4b involved 2711 nodes, 832 8-noded quadrilateral elements, and 48 focused quarter-point 6-noded triangular elements.

Table 1 presents the SIF values obtained by the proposed modified QP element in conjunction with DCT. Table 1 also contains the solution obtained by the MCCI, and the standard QP element in conjunction with DCT. As can be seen from Table 1 the solution obtained by proposed modified QP element agrees well with the reference solution obtained by the J-integral method.

A finer mesh needs to be employed when the standard QP elements are used, but in case of the modified QP elements the solution becomes less accurate as the mesh becomes finer. Thus better prediction of SIFs can be obtained with coarse fine when the modified QP elements are adopted. This may be explained by the fact that the crack tip elements must be long enough in order for the \( t^3 \) terms in the shape functions, \( \psi_2, \psi_4 \) to exhibit their presence.

**Example 2: Plate with the crack initiating from the central hole**

The example involves a flat-plate with a central hole subjected to far-field normal stress \( \sigma = 10000 \) units as shown in Fig. 5. The radius of the hole is 0.5 units, the thickness of the specimen is 0.04 units and the crack length is \( a = 0.10 \) units. Since it is a symmetric problem, only half of the model is modeled for analysis purpose. Both symmetric and displacement boundary conditions are applied.

![Plate with the crack initiating from the central hole](image)

Table 2 presents the SIF value, \( K_I \) obtained by the proposed modified QP element in conjunction with DCT. Table 2 also contains the solution obtained by the MCCI, and the standard QP element in conjunction with DCT. As can be seen from Table 2 the solution obtained by proposed modified QP element agrees well with the reference solution obtained by the J-integral method.

**Example 3: Interior cracked plate**

Consider a plate containing a single interior crack which is oriented arbitrarily with respect to the angle \( \theta \) as illustrated in Fig. 6. The dimensions of the plate are \( 2H = 2W = 200 \) units with a crack length of \( 2a = 0.4 \) units placed centrally. Two cases are analyzed.
In Case 1 the angle $\theta = 0$ (mode-I problem) and whereas in Case 2 the angle $\theta$ is varied between 0 and $\pi/2$ (mixed mode problem). In the finite element mesh eight elements are employed along the crack-length and rosette of 8 elements is taken around the crack tip.

**Case 1: $\theta = 0$ (Mode-I)**

Table 3 contains the SIFs for an infinite plate with a central crack ($\theta = 0$) at both left and right tips. The solution obtained by the $J$-integral method is taken as reference. With the mesh adopted, while the modified QP element can give very accurate value of the mode-I SIF with maximum % error = 0.3621, the result obtained from the MCCI has maximum % error = 2.8607 and the standard QP element has maximum % error = 0.9580.

**Mode –II Case ($0 < \theta < \pi/2$)**

Consider mixed-mode situation where ($0 < \theta < \pi/2$).

For this case the exact closed form solutions of the SIFs are available and are given by

\[ K_I^{Exact} = \sigma \sqrt{\pi a \sin^2 \theta} \]
\[ K_{II}^{Exact} = \sigma \sqrt{\pi a \sin \theta \cos \theta} \]

… (12)

Tables 4 and 5 presents respectively the normalized SIFs, $K_I/K_{I, Exact}$ and $K_{II}/K_{II, Exact}$ obtained by the proposed modified QP element and the Standard QP element in conjunction with DCT, as a function of $\theta$ at the right crack tip and at the left crack tip. Very accurate results of SIF were obtained by the proposed modified QP element.

**Example 4: Pair of interacting collinear cracks**

This is an example of two interacting cracks which have the same length $2a = 0.2$ units and are separated by a gap $b$ units (see Figure 7). The plate can be considered as an infinite domain by using the same dimensions as those in the previous example. The cracks are subjected to uniaxial tension $\sigma = 100$ applied in the direction perpendicular to the cracks. The mode-I SIF at the inner crack tips is of interest. In the finite element mesh, for each crack, eight

---

Table 3 — Comparison of SIFs for an infinite plate with a central crack ($\theta = 0$) at both left and right tips

<table>
<thead>
<tr>
<th>Method</th>
<th>$K_I^{Left}$ Predicted value</th>
<th>$K_I^{Left}$ Error (%)</th>
<th>$K_I^{Right}$ Predicted value</th>
<th>$K_I^{Right}$ Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MCCI</td>
<td>78.6327</td>
<td>0.9580</td>
<td>78.6331</td>
<td>0.9450</td>
</tr>
<tr>
<td>Standard QP element in conjunction with DCT</td>
<td>81.6143</td>
<td>2.8607</td>
<td>81.5679</td>
<td>2.7851</td>
</tr>
<tr>
<td>Modified QP element in conjunction with DCT</td>
<td>79.1174</td>
<td>0.2931</td>
<td>79.0711</td>
<td>0.3621</td>
</tr>
</tbody>
</table>

* $K_I^{Left} = 79.3524$, $K_I^{Left} = 79.2156$ obtained by $J$-Integral method is taken as reference

Table 4 — Normalized SIFs as a function of $\theta$ at the right crack tip

<table>
<thead>
<tr>
<th>Angle $\theta$ (degrees)</th>
<th>$K_I/K_{I, Exact}$ Standard QP element in conjunction with DCT</th>
<th>$K_I/K_{I, Exact}$ Modified QP element in conjunction with DCT</th>
<th>$K_{II}/K_{II, Exact}$ Standard QP element in conjunction with DCT</th>
<th>$K_{II}/K_{II, Exact}$ Modified QP element in conjunction with DCT</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>1.0132</td>
<td>1.0090</td>
<td>0.9620</td>
<td>0.9742</td>
</tr>
<tr>
<td>30</td>
<td>1.0145</td>
<td>1.0093</td>
<td>0.9634</td>
<td>0.9759</td>
</tr>
<tr>
<td>45</td>
<td>1.0162</td>
<td>1.0112</td>
<td>0.9692</td>
<td>0.9816</td>
</tr>
<tr>
<td>60</td>
<td>1.0148</td>
<td>1.0096</td>
<td>0.9660</td>
<td>0.9783</td>
</tr>
<tr>
<td>75</td>
<td>1.0138</td>
<td>1.0087</td>
<td>0.9668</td>
<td>0.9796</td>
</tr>
</tbody>
</table>
Table 5 — Normalized SIFs as a function of $\theta$ at the left crack tip

<table>
<thead>
<tr>
<th>Angle $\theta$ (degrees)</th>
<th>$K_{I}/K_{I\text{Exact}}$ Standard QP element in conjunction with DCT</th>
<th>$K_{II}/K_{II\text{Exact}}$ Standard QP element in conjunction with DCT</th>
<th>$K_{I}/K_{I\text{Exact}}$ Modified QP element in conjunction with DCT</th>
<th>$K_{II}/K_{II\text{Exact}}$ Modified QP element in conjunction with DCT</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>1.0126</td>
<td>0.9650</td>
<td>0.9638</td>
<td>0.9776</td>
</tr>
<tr>
<td>30</td>
<td>1.0130</td>
<td>1.0093</td>
<td>0.9692</td>
<td>0.9762</td>
</tr>
<tr>
<td>45</td>
<td>1.0163</td>
<td>1.0112</td>
<td>0.9655</td>
<td>0.9792</td>
</tr>
<tr>
<td>60</td>
<td>1.0137</td>
<td>1.0096</td>
<td>0.9665</td>
<td>0.9779</td>
</tr>
<tr>
<td>75</td>
<td>1.0130</td>
<td>1.0087</td>
<td>0.9663</td>
<td>0.9792</td>
</tr>
</tbody>
</table>

The effect of crack interaction is depicted in Fig. 8. The numerical error of $K_I$ increases as the gap $b$ between the cracks decreases. However, it can be seen that the increase in error rate for the standard QP is higher as the cracks come closer to each other. Moreover, the error from the modified QP is always much lower no matter what the value of the gap. This error is less than 1% even when the ratio $b/a$ is less than one-quarter.

Table 6 presents the comparison of SIF values for the two crack tips of interest for the two interacting collinear cracks for (i) $b/a = 1$, (ii) $b/a = 2$, (iii) $b/a = 3$, and (iii) $b/a = 4$. Table 6 contains the solution obtained by the proposed modified QP element in conjunction with DCT, the MCCI, and the standard QP element in conjunction with DCT. As can be seen from Table 6 the solution obtained by proposed modified QP element agrees well with the reference solution obtained by the $J$-integral method. Figures 9a and 9b depicts respectively the error (%) of the solution obtained by the three methods with reference to solution obtained by the $J$-integral method at left crack (right tip) and at right crack (left tip). As can be seen from Figs 9a and 9b error (%) with reference to solution obtained by the $J$-integral method is least for the proposed method when compared with the solution obtained by the MCCI, and the standard QP element.

Example 5: plate with off-centre crack

This is an example of off-centre crack and it deals with a mixed mode case in a finite body. Consider a crack whose location and orientation are arbitrary in a plate, subjected to remote uniaxial tension $\sigma = 1$ applied along the y-axis as shown in the Fig. 10. The following data are studied $2H = 2W = 2$ units, $2a = 0.5$ units. The centre of location of the crack is at a distance $e_x = e_y = 0.5$ units from the origin. The standard and modified QP shape functions are used with eight elements along each of the crack-length and rosette of 8 elements around each of the crack tip. Tables 7 and 8 presents respectively the normalized SIFs obtained by the proposed modified QP element, the standard QP element in conjunction with DCT and the $J$-integral method, as a function of $\theta$ at the crack
Table 6 — Comparison of SIF values for the two crack tips of interest for the two interacting collinear cracks;
(a) $b/a=1$, (b) $b/a=2$, (c) $b/a=3$, and (d) $b/a=4$

(a) Case 1: $b/a = 1$

<table>
<thead>
<tr>
<th>Method</th>
<th>$K_I$ Predicted (right tip)</th>
<th>Error (%)</th>
<th>$K_I$ Predicted (left tip)</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MCCI</td>
<td>61.6176</td>
<td>0.6904</td>
<td>61.5171</td>
<td>0.6794</td>
</tr>
<tr>
<td>Standard QP element in conjunction with DCT</td>
<td>63.2997</td>
<td>3.0511</td>
<td>63.3511</td>
<td>3.853</td>
</tr>
<tr>
<td>Modified QP element in conjunction with DCT</td>
<td>62.3135</td>
<td>0.2562</td>
<td>61.6911</td>
<td>0.2621</td>
</tr>
<tr>
<td>J-integral (Reference)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Case 2: $b/a = 2$

<table>
<thead>
<tr>
<th>Method</th>
<th>$K_I$ Predicted (right tip)</th>
<th>Error (%)</th>
<th>$K_I$ Predicted (left tip)</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MCCI</td>
<td>58.3136</td>
<td>0.6981</td>
<td>58.2691</td>
<td>0.7045</td>
</tr>
<tr>
<td>Standard QP element in conjunction with DCT</td>
<td>59.7123</td>
<td>2.5832</td>
<td>57.7415</td>
<td>2.0504</td>
</tr>
<tr>
<td>Modified QP element in conjunction with DCT</td>
<td>58.5156</td>
<td>0.2550</td>
<td>58.4539</td>
<td>0.2610</td>
</tr>
<tr>
<td>J-integral (Reference)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(c) Case 3: $b/a = 3$

<table>
<thead>
<tr>
<th>Method</th>
<th>$K_I$ Predicted (right tip)</th>
<th>Error (%)</th>
<th>$K_I$ Predicted (left tip)</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MCCI</td>
<td>57.2136</td>
<td>0.8673</td>
<td>57.1129</td>
<td>0.6442</td>
</tr>
<tr>
<td>Standard QP element in conjunction with DCT</td>
<td>59.2105</td>
<td>2.2424</td>
<td>58.6097</td>
<td>2.0097</td>
</tr>
<tr>
<td>Modified QP element in conjunction with DCT</td>
<td>57.5571</td>
<td>0.2487</td>
<td>57.3541</td>
<td>0.2608</td>
</tr>
<tr>
<td>J-integral (Reference)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(d) Case 4: $b/a = 4$

<table>
<thead>
<tr>
<th>Method</th>
<th>$K_I$ Predicted (right tip)</th>
<th>Error (%)</th>
<th>$K_I$ Predicted (left tip)</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MCCI</td>
<td>56.5137</td>
<td>0.7030</td>
<td>56.7461</td>
<td>0.8051</td>
</tr>
<tr>
<td>Standard QP element in conjunction with DCT</td>
<td>58.1127</td>
<td>2.0901</td>
<td>57.7441</td>
<td>1.1213</td>
</tr>
<tr>
<td>Modified QP element in conjunction with DCT</td>
<td>56.7613</td>
<td>0.2461</td>
<td>56.9981</td>
<td>0.2309</td>
</tr>
<tr>
<td>J-integral (Reference)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 9 — Error (%) with respect to $J$-integral; (a) error at left crack (right tip), (b) error at right crack (left tip)

Fig. 10 — Plate with off-centre crack tip A and at the crack tip B. It can be observed that the solutions obtained from the proposed modified QP element are much closer to reference solution obtained by $J$-integral method than the standard QP shape functions.
 improve the local COD solution at the crack tip, it is

are made with the existing analytical solutions

results. A similar story follows when the comparisons

functions, \(\hat{\psi}_2, \hat{\psi}_4\) and for it to have effect, the
element edge length \(L\) must be sufficient. Thus, a
relatively coarser mesh gives a far accurate result than
the standard QP element. From the results obtained
we can infer that the modified QP element can be
very useful for crack propagation simulations. It is
seen that not only for the Mode-I case, but also for the
Mixed-Mode case the use of the modified QP element
combined with the DCT technique gives far accurate
results when compared with standard \(J\)-integral\(^9\) results. A similar story follows when the comparisons
are made with the existing analytical solutions
existing. Since we have considered only linear elastic
fracture mechanics, the usage of the modified QP
element with modified shape functions only serves to
improve the local COD solution at the crack tip, it is
not expected to give accurate results for domain
integral method of SIF evaluation, e.g., \(J\)-integral.

Conclusions

It is clearly seen that with usage of the proposed
modified QP element, the numerical error in the SIFs
predicted by the DCT is reduced considerably when
compared to the solution predicted by the standard QP
element. The method requires a relatively coarse
mesh around the crack tip. The non-requirement of
very fine mesh around the crack tip can be accounted
for the fact that there is a \(r^3\) term in the new shape
functions, \(\hat{\psi}_2, \hat{\psi}_4\) and for it to have effect, the
element edge length \(L\) must be sufficient. Thus, a
relatively coarser mesh gives a far accurate result than
the standard QP element. From the results obtained
we can infer that the modified QP element can be
very useful for crack propagation simulations. It is
seen that not only for the Mode-I case, but also for the
Mixed-Mode case the use of the modified QP element
combined with the DCT technique gives far accurate
results when compared with standard \(J\)-integral\(^9\) results. A similar story follows when the comparisons
are made with the existing analytical solutions
existing. Since we have considered only linear elastic
fracture mechanics, the usage of the modified QP
element with modified shape functions only serves to
improve the local COD solution at the crack tip, it is

References


\begin{table}[h]
\centering
\caption{Normalized SIFs at crack tip a versus crack angle \(\theta\)}
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\multicolumn{3}{|c|}{Modified QP element} & \multicolumn{3}{|c|}{Standard QP element} \\
\multicolumn{3}{|c|}{in conjunction with} & \multicolumn{3}{|c|}{in conjunction with} \\
\multicolumn{3}{|c|}{DCT} & \multicolumn{3}{|c|}{DCT} \\
\hline
\(\theta\) & \(K_a^+/\sigma\sqrt{a}\) & \(K_a^+\) & \(K_a^+\) & \(K_b^+/\sigma\sqrt{a}\) & \(K_b^+\) & \(J\)-Integral \(J\) \\
\hline
0 & 1.2249 & 1.2501 & 1.2303 & 0.0262 & 0.0299 & 0.0276 \\
\(\pi/6\) & 0.8902 & 0.8937 & 0.8920 & 0.4750 & 0.4737 & 0.4887 \\
\(\pi/3\) & 0.3025 & 0.3009 & 0.3034 & 0.4622 & 0.4621 & 0.4742 \\
\(\pi/2\) & 0 & 0 & 0 & 0 & 0 & 0 \\
\(2\pi/3\) & 0.2765 & 0.2781 & 0.2789 & \(-0.4770\) & \(-0.4742\) & \(-0.4894\) \\
\(5\pi/6\) & 0.8792 & 0.8846 & 0.8832 & \(-0.5060\) & \(-0.4996\) & \(-0.5190\) \\
\(\pi\) & 1.2249 & 1.2536 & 1.2319 & \(-0.0412\) & \(-0.0446\) & \(-0.0431\) \\
\hline
\end{tabular}
\end{table}

\begin{table}[h]
\centering
\caption{Normalized SIFs at crack tip b versus crack angle \(\theta\)}
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\multicolumn{3}{|c|}{Modified QP element} & \multicolumn{3}{|c|}{Standard QP element} \\
\multicolumn{3}{|c|}{in conjunction with} & \multicolumn{3}{|c|}{in conjunction with} \\
\multicolumn{3}{|c|}{DCT} & \multicolumn{3}{|c|}{DCT} \\
\hline
\(\theta\) & \(K_b^+/\sigma\sqrt{a}\) & \(K_b^+\) & \(K_b^+\) & \(K_b^+/\sigma\sqrt{a}\) & \(K_b^+\) & \(J\)-Integral \(J\) \\
\hline
0 & 1.2009 & 1.2345 & 1.2043 & \(-0.0412\) & \(-0.0441\) & \(-0.0437\) \\
\(\pi/6\) & 0.9427 & 0.9465 & 0.9446 & 0.4730 & 0.4674 & 0.4851 \\
\(\pi/3\) & 0.3077 & 0.3090 & 0.3084 & 0.4822 & 0.4762 & 0.4944 \\
\(\pi/2\) & 0 & 0 & 0 & 0 & 0 & 0 \\
\(2\pi/3\) & 0.2765 & 0.2787 & 0.2787 & \(-0.4770\) & \(-0.4741\) & \(-0.4890\) \\
\(5\pi/6\) & 0.8792 & 0.8841 & 0.8833 & \(-0.5060\) & \(-0.4990\) & \(-0.5191\) \\
\(\pi\) & 1.2249 & 1.2501 & 1.2345 & 0.0262 & 0.0297 & 0.0276 \\
\hline
\end{tabular}
\end{table}