Air entrainment by breaking waves: A theoretical study

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The variation of static pressure, density, potential energy and kinetic energy due to air bubble entrainment in the surf zone and their physical mechanisms have been studied theoretically. An averaging technique has been used to investigate the behavior of every single bubble in the bubble cloud. The above parameters have been found to be affected significantly by the entrained air bubbles.

[Key words: Void fraction, pressure, density, potential energy, kinetic energy, air entrainment, waves breaking, breaking waves]

1. Introduction

Wave breaking is a natural phenomenon that is widespread in the nearshore regions. It induces a modification of wave shapes and contains a large number of air bubbles in water, resulting in complicated flow field of air-water. It is unclear how these air bubbles affect the fluid motion in the surf zone. Recently Chanson et al.\(^1\) suggested that wave breaking near the coastline associated with significant sediment transport and resulting flow becomes in three-phases: air (gas), water (liquid) and sediment (solid). Owing to the complicated phenomena of air-water mixture, surf zone hydrodynamics presents considerable difficulties for both experiments and numerical modeling. Hence, the study on surf zone air bubbles is limited.

Air entrainment is associated with a rise in water level caused by the liquid displacement upwards. Führböter\(^2\) suggested that the entrained air causes a transfer of energy into potential energy. Hoque\(^3\) proposed that the sudden reduction of wave height and energy dissipation inside the surf zone can be explained partially by the entrained of air bubbles into water. Some researchers showed experimentally that the void fraction distributions consistently decay exponentially with the water depth\(^4,6\). Deane & Stokes\(^7\) measured the bubble size distributions and bubble formation mechanisms using optical method. Hwang et al.\(^8\) investigated the air bubbles by breaking waves in a laboratory experiments, and found that the wave height appeared to be the appropriate scaling length for the vertical distribution of void fraction and their horizontal distribution correlated well with the group characteristics of waves. Graham\(^9\) estimated the volume of entrained air and the rate of transfer of carbon dioxide between bubbles and water in the mixture.

The above works have been done considering air bubble effects, but virtually all of the investigations were experimental studies. This study investigates basic air entrainment characteristics in the surf zone. In order to give an overall picture of the dynamical role of air bubbles (Fig. 1), an averaging technique is used to describe the influence of the entrained air bubbles. The results presents new evidence leading to a better understanding of air bubble effects on the wave parameters.

2. Theory and Experiments

2.1 Vertical distribution of void fraction

With respect to the modeling of void fraction profiles under breaking waves, it is possible to treat the cases of both spilling and plunging breakers in the...
same manner. In the present study the distribution of air bubbles in the vertical direction is considered to take the following exponential form proposed by Wu:\(^{10}\)

\[
C(z) = C_0 \exp(k_1z)
\]  

where \(C(z)\) is the part of the volume locally occupied by air bubbles per unit width (time-averaged concentration), \(k_1\) is a decay parameter characterizing vertical distribution of air bubbles and \(C_0\) denotes the reference void fraction at the mean water surface \(z = 0\).

The following boundary conditions are automatically satisfied:

\(C(z) = C_0\) at the surface \(z = 0\)

And \(C(z) \rightarrow 0\), for \(z \rightarrow -\infty\)

2.2 Water level rise

The rise of the free-surface level \(\Delta h\) is a function of the amount of entrained air and water depth (Fig. 2). The total volume of entrained air into water per unit width is defined as

\[
\Delta h = \int_{-\Delta h}^{0} C(z) \, dz
\]  

where \(z\) is taken upward from the raised water surface.

Now Eq. (2a) can be solved with the help of Eq. (1) in the following manner:

\[
\Delta h = \frac{C_0}{k_1} \left[1 - e^{-k_1\Delta h} \left[1 - k_1\Delta h + \frac{(k_1\Delta h)^2}{2!} - \cdots \right]\right]
\]

Neglecting the higher term of \(\Delta h\), we can write in explicit and dimensionless form of the above equation:

\[
\frac{\Delta h}{h} = \frac{C_0}{k_1 h} \left(1 - e^{-k_1\Delta h} \right)\]

\[\cdots \text{(2b)}\]

2.3 Averaging procedure

It is mentioned earlier that, a great amount of air bubble is entrained at the breaking point, but it is impossible to investigate the behavior of every single bubble in the bubble cloud (Fig. 1). By applying an averaging procedure\(^{11,12}\), all quantities like water pressure and bubble radius become continuous functions of space rather than discrete. According to Biesheuvel & van Wijngaarden\(^{12}\), the averaged quantities such as, pressure \(p\), horizontal velocity \(u\), vertical velocity \(w\), and density \(\rho\), where the averaging is taken over the mixture containing many bubbles can be summarized as follows:

\[
u = (1 - C)u_w + Cu_a\]  

\[\cdots \text{(3)}\]

\[
w = (1 - C)w_w + C(w_a + w_r)\]  

\[\cdots \text{(4)}\]

\[
p = (1 - C)p_w + Cp_a - \frac{2\sigma}{R} C\]  

\[\cdots \text{(5)}\]

\[
\rho = (1 - C)p_w + C\rho_a\]  

\[\cdots \text{(6)}\]

where, subscripts ‘a’, ‘w’ denote the air and water, respectively and \(w_r\) denotes the rise velocity of bubbles. For the case of radial motion of a bubble, the internal pressure of air bubble can be expressed by

\[
p_a = p_w + \frac{2\sigma}{R}\]  

\[\cdots \text{(7)}\]

where \(\sigma\) and \(R\) denote surface tension and air bubble radius, respectively.

Density of the aerated region will generally be less than that of the undisturbed seawater owing to entrained air. Since the air density is much smaller
than the water density and it can be neglected. If the surface tension of air bubble is left out of account and using Eq. (7), the above relations reduce to:

\[
\begin{align*}
    u &= (1 - C)u_w + Cu_a & \ldots (8) \\
    w &= (1 - C)w_w + C(w_a + w_r) & \ldots (9) \\
    p &= p_w & \ldots (10) \\
    \rho &= (1 - C)\rho_w & \ldots (11)
\end{align*}
\]

Under further assumption that the horizontal and vertical velocity fields of the water do not change significantly due to the air bubbles entrainment, i.e. \( u_w = u_a \) and \( w_w = w_a \), then above relations are approximated as

\[
\begin{align*}
    u &= u_w \\
    w &= w_w + w' \\
    p &= p_w \\
    \rho &= (1 - C)\rho_w
\end{align*}
\]

2.4 Correction term and boundary condition

The above assumptions do not satisfy the continuity equation. Note that the term \( Cw_r \) is not the time dependent quantities, which might be the reason for the discontinuity. Requiring now that to satisfy the continuity equation, we therefore modify the vertical velocity term in the following manner:

\[
\begin{align*}
    w &= w_w + w' & \ldots (13)
\end{align*}
\]

where \( w' \) is the correction term.

For steady flow, the continuity equation becomes

\[
\begin{align*}
    \text{div}(\rho \nabla) &= 0 & \ldots (14) \\
    \text{i.e. } \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho w)}{\partial z} &= 0 & \ldots (15)
\end{align*}
\]

where \( \rho \) is the function of \( z \) only.

Finally, the first order linear differential equation in terms of \( w' \) can be obtained with the help of \( u, w \) and continuity equation:

\[
\begin{align*}
    \frac{\partial w'}{\partial z} - \frac{C_0k_1e^{kz}}{(1 - C_0e^{kz})} w' &= \frac{C_0k_1e^{kz}}{(1 - C_0e^{kz})} w_w & \ldots (16)
\end{align*}
\]

which is subject to the boundary condition of vanishing the correction term at the bottom i.e., \( w' \rightarrow 0 \) for \( z = -h \).

Hence, the final form of correction term can be expressed by

\[
\begin{align*}
    w' &= (-\frac{\pi H}{T \sinh kh}) \frac{C_0}{(1 - C_0e^{kz})} \left[ e^{kz} \left( \frac{kk}{k^2 - k_1^2} \cosh(k(h + z)) - \frac{kk}{k^2 - k_1^2} e^{-kh} \right) \sin(\omega t - kx) \right] & \ldots (17)
\end{align*}
\]

2.5 Determination of \( C_0 \) and \( k_1 \)

In Eq. (1), the decay parameter \( k_1 \) and the reference void fraction \( C_0 \) can be estimated by comparison with experimental data. The void fraction \( C \) is the proportion of time that the probe tip is in the air. The measurements of void fraction in the surf zone were carried out by Hoque & Aoki. The parameter \( k_1 \) for void fraction distribution in the surf zone was determined by fitting a theoretical curve to the experimental data only for plunging breakers. \( k_1 \) has found smaller value when penetration is larger and vice-versa. On the other hand, to find a reasonable value or expression of \( C_0 \), three sets of data were used for plunging breakers. The parameters \( C_0 \) was calculated experimentally from the wave tests in the flume. The best-fit curves to the values of \( C_0 \) are found which depend on the horizontal distance from the breaking point \((x-x_b)/L_0\).

All the data for void fraction \( C_0 \) are corrected by the following expressions for plunging breakers:

\[
\begin{align*}
    C_0 &= 1.285 \times \frac{(x-x_b)}{L_0} \text{ for } 0 \leq \frac{(x-x_b)}{L_0} \leq 0.14 \\
    C_0 &= -0.75 \times \frac{(x-x_b)}{L_0} + 0.285 \text{ for } 0.14 \leq \frac{(x-x_b)}{L_0}
\end{align*}
\]

3. Evaluation of Wave Parameters

3.1 Density and pressure field

Under the wave conditions, density and pressure fields in the vertical direction are determined by the local void fraction. Because of air bubbles entrained, the air water mixture becomes significantly lighter than the water below it. The vertical structure of density (Fig. 3A) is determined using Eq. (18) for different \( C_0 \) and wave decay factor \( k_1 \).

\[
\rho = (1 - C_0 \exp(k_1z)) \rho_w & \ldots (18)
\]

On the other hand, the pressure force is assumed to be hydrostatic in the absence of waves. The static pressure includes the effect of water level rise by entrained air but is different from triangular distribution (Fig. 3B). The static pressure including water level rise by air bubbles is given by:
\[ p(z) = p_{atm} - \rho_w g \left\{ z + \frac{C_0}{k_1} (1 - e^{k_1 z}) \right\} \]  \hspace{1cm} (19)

where \( p_{atm} \) is the atmospheric pressure and the above relation satisfies the boundary conditions that \( p = p_{atm} \) at \( z = 0 \); \( p = p_{atm} + \rho_w gh \) at \( z = -h - \Delta h \).

### 3.2 Static energy

According to Führböter\textsuperscript{2} and based on Fig. 2, the excess static energy per unit horizontal area due to entrained air bubbles inside the surf zone can be expressed by

\[
\Delta SE = SE - SE_0 = \int_{-h-\Delta h}^{0} \rho_w g z \, dz - \int_{-h-\Delta h}^{-h} \rho_w g z \, dz
\]

The first term on the right hand side in the above equation can be recognized as total potential energy including air bubble effect and second term without air bubbles. It is given in a dimensionless form:

\[
M = \frac{\Delta SE}{\rho_w gh^2/2} = C_0 \left[ \frac{2(1 - e^{-k_1 h})}{(k_1 h)^2} - \frac{2(1 - C_0)}{k_1 h} \frac{e^{-k_1 h}}{(1 - C_0 e^{-k_1 h})} \right]
\]  \hspace{1cm} (20)

This is only the static energy which is necessary to transfer the water height and air volume \( \Delta h \) into \( h + \Delta h \). Figure 4 shows dependence of \( M \) on \( C_0 \) and \( k_1 h \) which suggests that the static energy increases with increasing \( C_0 \) and decreasing \( k_1 h \).

### 3.3 Potential energy

Since air entrainment is associated with a rise in water level caused by the liquid displacement, resulting in potential energy increases. The increased potential energy \( \Delta PE \) due to air bubbles is defined as the difference between the potential energy of water...
with air bubbles (second bracket term) and without air bubbles in presence of waves:

$$\Delta P_E = PE - PE_0 = \left\{ \int_{h - \Delta h}^{h} \rho_w (1 - C_0) g \, dz \right\} - \left\{ \int_{h - \Delta h}^{h} \rho_w g \, dz \right\} - \int_{0}^{\Delta h} \rho_w g \, dz - \int_{0}^{\Delta h} \rho_w g \, dz$$

... (21a)

Simplifying,

$$\Delta P_E = \frac{1}{T} \int_{0}^{T} \int_{h - \Delta h}^{h} \rho_w (1 - C_0) g \, dz \, dt + \frac{1}{T} \int_{0}^{T} \int_{0}^{\Delta h} \rho_w g \, dz \, dt$$

After integration, we have

$$\Delta P_E = \frac{\rho_w g}{2} \left( \frac{(h + \Delta h)^2}{k_1^2} \cdot \frac{C_0}{k_1} \cdot \frac{e^{-k_1 h}}{2!} \right) + \frac{\rho_w g}{T} \left\{ \int_{0}^{\eta} \left( \frac{\eta^3}{2} - \frac{C_0}{k_1 \eta} - \frac{C_0}{2!} \eta^2 \right) \, dt \right\} + \frac{\rho_w g}{2} \left( \frac{(h + \Delta h)^2}{k_1^2} \cdot \frac{C_0}{k_1} \cdot \frac{e^{-k_1 h}}{2!} \right) + \frac{\rho_w g}{T} \left\{ \int_{0}^{\eta} \left( \frac{\eta^3}{2} - \frac{C_0}{k_1 \eta} - \frac{C_0}{2!} \eta^2 \right) \, dt \right\}$$

Neglecting the higher order terms of \( \Delta h, \eta \) and using \( \frac{1}{T} \int_{0}^{T} \eta \, dt = \bar{\eta} \) which gives

$$\Delta P_E = \frac{\rho_w g}{2} \left( \frac{\eta^3}{2} \cdot \frac{1}{k_1^2 h} \cdot \frac{e^{-k_1 h}}{2!} \right) + \frac{\rho_w g}{T} \left\{ \int_{0}^{\eta} \left( \frac{\eta^3}{2} - \frac{C_0}{k_1 \eta} - \frac{C_0}{2!} \eta^2 \right) \, dt \right\}$$

The above equation can be expressed in non-dimensional form:

$$N = \frac{\Delta P_E}{\rho_w g h^2 / 16}$$

$$= C_0 [-1 + 16 \left( \frac{1 - e^{-k_1 h}}{k_1 h^2} \right) \frac{(1 - C_0)}{(k_1 h^2) - (1 - C_0) e^{-k_1 h}}]$$

... (21b)

where \( \eta^2 = h^2 / 8 \) for sinusoidal waves. Figure 5 illustrates the relation between \( N \) and \( C_0, k_1 h \) and \( k_1 H \) which is calculated from Eq. (21b). The dimensionless potential energy \( N \) increases as \( C_0 \) increasing and \( k_1 H \) decreasing. The dimensional potential energy also shows a significant change for \( k_1 H < 2 \) (Fig. 5D). Figure 5C suggests that \( N \) becomes flat for \( k_1 h > 5 \) and decreases for \( k_1 h < 5 \) for a certain value of \( k_1 H = 3.0 \).

![Fig. 5](image-url) — Dimensionless potential energy as a function of (A) \( C_0 \) (where \( k_1 H \) is constant), (B) \( C_0 \) (where \( k_1 h \) is constant), (C) \( k_1 h \) and (D) \( k_1 H \).
3.4 Kinetic energy

The kinetic energy per unit surface area is obtained by integrating that per unit volume over the depth and averaging it over the wave period:

\[ KE' = \int_{-h}^{0} \frac{1}{2} \rho (u^2 + w^2) \, dz \]

Here, the \( z \)-integral is taken up to the mean water level: \( z = 0 \), because the integral up to \( z = \eta \) gives a higher order term. For the surface waves on water with air bubbles the kinetic energy \( KE' \) is estimated as

\[ KE' = \int_{-h}^{0} \frac{1}{2} \rho_a (1-C)(u^2 + (w + w')^2) \, dz \]

Since simplifying the above expression in analytical form is difficult, a numerical procedure is introduced. In dimensionless form of kinetic energy:

\[ O = \frac{KE'}{\rho_a g h^2 / 16} = \frac{2k}{\sinh 2kh} \int_{-h}^{0} \left[ (1-C_0 e^{kh}) \cosh^2 k(h+z) 
+ \frac{C_0}{(1-C_0 e^{kh})(k^2 - k_1^2)} \left\{ e^{kh} (kk_1 \cosh k(h+z) + k_1^2 \sinh k(h+z)) - kk_1 e^{-kh} \right\} \right] \, dz \]

\[
\ldots (22)
\]

Figure 6 shows the dimensionless kinetic energy represented by Eq. (22). In the presence of waves the kinetic energy decreases with air bubble increasing into water (Fig. 6A), but this effect is almost negligible in comparison with the increase in potential energy. Comparing Figs. 5 A, B with Fig. 6 A, B, it is found that the kinetic energy decreases 2% by air bubbles, whereas the increase in potential energy is around 15% for \( C_0 = 0.20 \) and \( k_1 h = 0.6 \). It is also seen (Fig. 6C) that air bubbles effect is almost constant for \( kh > 1.2 \) and this range may be deep-water wave conditions. Note that the limits of three regions shallow water, intermediate depth, and the deep water are denoted by \( kh < \pi / 10, \pi / 10 < kh < \pi, \) and \( kh > \pi, \) respectively. In the range of \( kh < 1.2, \) Fig. 6C shows dimensionless kinetic energy is affected by the entrained air significantly.

4. Conclusion

This paper has been focused on some properties of wave parameters theoretically in terms of air bubble effects. The variation of wave parameters (e.g. density, static pressure, potential energy and kinetic energy) has been found significantly due to air bubble
entrainment. In Figs. 3, 6, the numeric values of $C_0$ and $k_1$ are chosen randomly. Two parameters $C_0$ and $k_1$ used in Eq. (1) were determined by comparing with the experimental results. Equations (2b) and (20) are compared successfully with the experimental data in horizontal directions (Fig. 7). Note that Eqs. (21b) and (22) cannot be verified because we don’t have the experimental data of wave height $H$. In Fig. 7, the variations of air volume and static energy can be divided into three stages in characteristics. The air volume and static energy rapidly decrease before and after the impinging point of broken waves. The present study shows that the distribution of static energy (Fig. 7) is almost similar with the study of Hwung et al.  

In summary, the study contributes to a better understanding of increased static, potential and kinetic energy due to air bubbles entrainment. It is believed the air bubbles in breaking waves play important role to dissipate the wave energy. Future studies to investigate wave energy dissipation by surf zone air bubbles both in quantitatively and qualitatively are in progress.

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References


