Energy balance method and amplitude frequency formulation based simulation of strongly non-linear oscillators

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The Energy Balance Method (EBM) and Amplitude Frequency Formulation (AFF) have been applied to derive the approximate analytical solution for motion of two mechanical oscillators. In first problem both methods yield the same result but in the second problem, the results which are obtained from Amplitude Frequency Formulation and Energy Balance Method are different. In comparison with the forth order Runge-Kutta method, the results show that these methods are very convenient for solving non-linear equations and also can be used for strong non-linear oscillators.

Keywords: Non-linear oscillation, Energy balance method, Amplitude frequency formulation, Analytical solution

1 Introduction

In the present paper, the motion equations of two oscillators by Energy Balance Method and Amplitude Frequency Formulation to obtain the relationship between amplitude and angular frequency, have been investigated. As we see in the geometry of problem (Fig. 1), \( m_1 \) is mass of the block on the horizontal surface, \( m_2 \) the mass of block which is just slipped in the vertical and is linked to \( m_1 \), \( L \) the length of link, \( g \) the gravitational acceleration and \( k \) is spring constant.

By assuming \( u = \frac{x}{L} \ll 1 \), the equation of motion can yield as in the following terms:

\[
\ddot{u} + \left( \frac{m_2}{m_1} \right) u \ddot{u} + \left( \frac{m_2}{m_1} \right) \dot{u}^2 + \left( \frac{k}{m_1} + \frac{m_2 g}{l m_1} \right) u + \frac{m_2 g}{2 l m_1} u^3 = 0
\]

In which \( u \) and \( t \) are generalized dimensionless displacements and time variables, and \( \Lambda = \frac{\Omega^2 r}{g} \).

In recent years, many powerful methods have been used to find approximate solution to the non-linear differential equations. Some of these methods are Homotopy Perturbation Method¹-⁴ (HPM), Max-Min Approach⁵,⁶ (MMA), Variational Iteration Method⁷-⁹ (VIM), Energy Balance Method¹⁰-¹³ (EBM), Amplitude Frequency Formulation¹⁴-¹⁶ (AFF) and Adomian Decomposition Method¹⁷ (ADM). Other approximations can also be mentioned in this case¹⁸-²⁰.

In which \( \theta \) and \( t \) are generalized dimensionless displacements and time variables, and \( \Lambda = \frac{\Omega^2 r}{g} \).

![Fig. 1 — Geometry of first problem](image-url)
In the present paper, the problems which are strongly non-linear by Energy Balance Method and Amplitude Frequency Formulation, have been investigated. The comparison between approximate solutions and the forth-order Runge-Kutta method assures us about accuracy and validity of solving.

2 Problem Formulation

2.1 Energy Balance Method

The solution of first problem with the Energy Balance Method is:

$$\dddot{u} + \left( \frac{m_2}{m_1} \right) u^2 \dddot{u} + \frac{m_2 g}{2l m_1} u^3 + \alpha_0^2 u = \frac{m_2 g}{2l m_1} u^3 = 0, u(0) = A, \dot{u}(0) = 0$$

(3)

where

$$\alpha_0^2 = \left( \frac{k}{m_1} + \frac{m_2 g}{l m_1} \right)$$

Its variational formulation can be easily established:

$$J(u) = \int_{0}^{T} \left( -\frac{1}{2} \dddot{u}^2 + \left( \frac{m_2}{2m_1} \right) u^2 \dddot{u}^2 + \frac{m_2 g}{8l m_1} u^4 \right) dt$$

(4)

The Hamilton of Eq. (1), therefore, can be written in the form:

$$H = \frac{1}{2} \dddot{u}^2 + \left( \frac{m_2}{2m_1} \right) u^2 \dddot{u}^2 + \frac{1}{2} \alpha_0^2 u^2 + \frac{m_2 g}{8l m_1} u^4$$

$$= \frac{1}{2} \alpha_0^2 A^2 + \frac{m_2 g}{8l m_1} A^4$$

(5)

or:

$$R(t) = \frac{1}{2} \dddot{u}^2 + \left( \frac{m_2}{2m_1} \right) u^2 \dddot{u}^2 + \frac{1}{2} \alpha_0^2 u^2 + \frac{m_2 g}{8l m_1} u^4$$

$$- \frac{1}{2} \alpha_0^2 A^2 - \frac{m_2 g}{8l m_1} A^4 = 0$$

(6)

We use the following initial approximate guess to determine the angular frequency:

$$u(t) = A \cos(\alpha t)$$

(7)

Substituting Eq. (7) into Eq. (6), we can obtain:

$$R(t) = \frac{1}{2} A^2 \alpha^2 \sin^2(\alpha t) + \left( \frac{m_2}{2m_1} \right) A^2 \alpha^2 \sin^2(\alpha t) \cos^2(\alpha t)$$

$$+ \frac{1}{2} \alpha_0^2 A^2 \cos^2(\alpha t) + \frac{m_2 g}{8l m_1} A^4 \cos^4(\alpha t)$$

$$- \frac{1}{2} \alpha_0^2 A^2 - \frac{m_2 g}{8l m_1} A^4 = 0$$

(8)

which trigger the following results:

$$\omega = \sqrt{\left( \frac{1}{2} \alpha_0^2 A^2 \cos^2(\alpha t) + \frac{m_2 g}{8l m_1} A^4 \cos^4(\alpha t) \right) - \left( -\frac{1}{2} \alpha_0^2 A^2 - \frac{m_2 g}{8l m_1} A^4 \right)}$$

$$\frac{1}{2} A^2 \sin^2(\alpha t) + \left( \frac{m_2}{2m_1} \right) A^4 \sin^2(\alpha t)$$

(9)

If we collocate at $\alpha t = \frac{\pi}{4}$, we obtain:

$$\omega_{EBM} = \frac{1}{2} \sqrt{\frac{8 \alpha_0^2 l m_1 + 3 m_2 g A^2}{l (2m_1 + m_2 A^2)}}$$

(10)

2.2 Amplitude Frequency Formulation

According to He's frequency formulation, we can assume for the Amplitude Frequency Formulation:
\[ \omega^2 = \frac{\omega_1^2 R_1(t_1) - \omega_2^2 R_1(t_1)}{R_1(t_2) - R_1(t_1)} \quad \ldots (11) \]

To solve Eq. (1), we first use two trial functions as follows:

\[ u_1 = A \cos(t) \quad \ldots (12) \]

\[ u_2 = A \cos(\omega t) \quad \ldots (13) \]

Inserting Eqs (11 and 12) into Eq. (1), gives the following Residuals:

\[ R_1(t_1) = -A \cos(t) - A^3 \left( \frac{m_2}{m_1} \right) \cos(t)^3 \]

\[ + A^3 \left( \frac{m_2}{m_1} \right) \cos(t) \sin(t)^2 + A \omega_1^2 \cos(t) \]

\[ + A^3 \left( \frac{m_2 g}{2lm_1} \right) \cos(t)^3 \quad \ldots (14) \]

\[ R_2(t) = -A \cos(\omega t) \omega^2 - A^3 \left( \frac{m_2}{m_1} \right) \cos(\omega t)^3 \omega^2 \]

\[ + A \omega_1^2 \cos(\omega t) + A^3 \left( \frac{m_2}{m_1} \right) \cos(\omega t) \sin(\omega t)^2 \omega^2 \]

\[ + A^3 \left( \frac{m_2 g}{2lm_1} \right) \cos(\omega t)^3 \quad \ldots (15) \]

Weighted residuals can be introduced as follows:

\[ \tilde{R}_1(t_1) = \frac{4}{T_1} \int_0^{T_1} R_1(t) \cos(t) dt, T_1 = \frac{2 \pi}{\omega_1} \quad \ldots (16) \]

\[ \tilde{R}_2(t_2) = \frac{4}{T_2} \int_0^{T_2} R_2(t) \cos(\omega t) dt, T_2 = \frac{2 \pi}{\omega_2} \quad \ldots (17) \]

Equating \( \omega_1 = 1, \omega_2 = \omega \), we can obtain weighted residuals:

\[ \tilde{R}_1(t_1) = \frac{2}{\pi} \int_0^{T_1} \left( -A \cos(t) - A^3 \left( \frac{m_2}{m_1} \right) \cos(t)^3 \right. \]

\[ + A^3 \left( \frac{m_2}{m_1} \right) \cos(t) \sin(t)^2 + A \omega_1^2 \cos(t) \]

\[ \left. + A^3 \left( \frac{m_2 g}{2lm_1} \right) \cos(t)^3 \right) \cos(t) dt = \frac{1}{2} A \omega_1^2 \]

\[ + \frac{3}{8} A^3 \left( \frac{m_2 g}{2lm_1} - \frac{1}{4} \left( \frac{m_2}{m_1} \right) A^3 - \frac{1}{2} A \quad \ldots (18) \]

\[ \tilde{R}_2(t_2) = \frac{2}{\pi} \int_0^{T_2} \left( -A \cos(\omega t) \omega^2 - A^3 \left( \frac{m_2}{m_1} \right) \cos(\omega t)^3 \omega^2 \right. \]

\[ + A \omega_1^2 \cos(\omega t) + A^3 \left( \frac{m_2}{m_1} \right) \cos(\omega t) \sin(\omega t)^2 \omega^2 \]

\[ \left. + A^3 \left( \frac{m_2 g}{2lm_1} \right) \cos(\omega t)^3 \right) \cos(\omega t) d\varphi \]

\[ = -\frac{1}{2} A \omega^2 + \frac{3}{8} \left( \frac{m_2 g}{2lm_1} \right) A^3 - \frac{1}{4} A^3 \left( \frac{m_2}{m_1} \right) \omega^2 \]

\[ + \frac{1}{2} A \omega_1^2 \quad \ldots (19) \]

Inserting Eqs (18 and 19) into Eq. (11), angular frequency can yield as:

\[ \frac{1}{2} A \omega_1^2 A + \frac{3}{8} \left( \frac{m_2 g}{2lm_1} \right) A^3 - \frac{1}{4} A^3 \left( \frac{m_2}{m_1} \right) \omega^2 \]

\[ - \frac{1}{2} A \omega^2 - \omega^2 \]

\[ - \frac{1}{4} A^3 \left( \frac{m_2}{m_1} \right) \omega^2 - \frac{1}{2} A \omega^2 + \frac{1}{4} \left( \frac{m_2}{m_1} \right) A^3 + \frac{1}{2} A \]

\[ \omega^2 = \frac{1}{2} A \omega_1^2 \omega_2^2 \quad \ldots (20) \]

Solving Eq. (20), Amplitude-frequency relationship can be obtained:

\[ \omega_{AFF} = \omega_{EBM} = \frac{1}{2} \left( \frac{8 \omega_1^2 \omega_2^2 + 3m_2 g A^2}{l (m_2 A^2 + 2m_1)} \right) \quad \ldots (21) \]

In Figs 3 and 4, the comparison between Analytical solutions and Runge-Kutta result is shown. As we see, Amplitude Frequency Formulation and Energy Balance Method obtain same results and have a high validity in comparison with Runge-Kutta method.

3 Results and Discussion

3.1. Energy Balance Method

Variational principle of Eq. (2) can be easily obtained:

\[ J(\theta) = \int \left( -\frac{1}{2} \dot{\theta}^2 - \cos(\theta) + \frac{A}{2} \cos^2(\theta) \right) d\theta \quad \ldots (22) \]
Fig. 3 — Comparison among amplitude frequency formulation, Energy Balance Method and Runge-Kutta fourth order in first problem for
\[ A = \frac{\pi}{6}, \quad g = 9.81m/s^2, \quad k = 300N/m^2, \quad m_1 = 10kg, \]
\[ m_2 = 2kg, \quad l = 0.5m \]

Its Hamilton, therefore, can be written in the form:
\[ H = \frac{1}{2} \dot{\theta}^2 - \cos(\theta) + \frac{\Lambda}{2} \cos^2(\theta) \]
\[ = -\cos(A) + \frac{\Lambda}{2} \cos^2(A) \] … (23)

Similar of previous example we have:
\[ R(t) = \frac{1}{2} A^2 \omega^2 \sin^2(\omega t) - \cos\left(A \cos(\omega t)\right) \]
\[ + \frac{\Lambda}{2} \cos^2\left(A \cos(\omega t)\right) + \cos A \]
\[ - \frac{\Lambda}{2} \cos^2 A = 0 \] … (24)

We obtain the following result:
\[ \omega = \frac{\sqrt{2}}{A} \sin(\omega t) \left[ \cos\left(A \omega - \cos(A)\right) \right] \]
\[ + \frac{\Lambda}{2} \left[ \cos^2 A - \cos^2\left(\frac{\sqrt{2}}{2} A\right)\right] \] … (25)

If we collected at \( \omega = \frac{\pi}{4} \), we obtain:
\[ \omega_{EBM} = \frac{2}{A} \left[ \cos\left(\frac{\sqrt{2}}{2} A\right) - \cos(A) \right] \]
\[ + \frac{\Lambda}{2} \left[ \cos^2 A - \cos^2\left(\frac{\sqrt{2}}{2} A\right)\right] \] … (26)

Therefore:
\[ \theta(t) = A \cos \left( \frac{2}{A} \left[ \cos\left(\frac{\sqrt{2}}{2} A\right) - \cos(A) \right] \right) t \]
\[ + \frac{\Lambda}{2} \left[ \cos^2 A - \cos^2\left(\frac{\sqrt{2}}{2} A\right)\right] \] … (27)

3.2 Amplitude Frequency Formulation
Substituting \( \theta \) mentioned in Eq. (27) into Eq. (2), can be rewritten as Eq. (28):
\[ \ddot{\theta} + \sin(\theta) - \frac{1}{2} A \sin(2\theta) = 0, \quad \dot{\theta}(0) = 0, \quad \theta(0) = A \] … (28)

Substitution of the relatively accurate approximations:
\[ \sin(\theta) = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} \quad \text{and} \quad \sin(2\theta) = 2\theta - \frac{4\theta^3}{3} + \frac{4\theta^5}{15} \] … (29)

into Eq. (27), yields:
\[ \ddot{\theta} + (1-\Lambda)\theta + \left( -\frac{1}{6} + \frac{2\Lambda}{3} \right)\dot{\theta}^3 + \left( \frac{1}{120} - \frac{2\Lambda}{15} \right)\dot{\theta}^5 = 0 \]

… (30)

The trial functions \( \dot{\theta}_1(t) = A\cos(\tau) \) and \( \dot{\theta}_2(t) = A\cos(\omega t) \) are inserted into Eq. (30) to yield the residuals:

\[
R_1(t) = - \frac{1}{6} A^3 \cos(t)^3 + \frac{1}{120} A^5 \cos(t)^5
\]
\[\quad + \frac{\Lambda}{2} \left( 2A\cos(t) - \frac{4}{3} A^3 \cos(t)^3 + \frac{4}{15} A^5 \cos(t)^5 \right) \]

… (31)

\[
R_2(t) = - A\cos(\omega t)\omega^2 + A\cos(\omega t) - \frac{1}{6} A^3 \cos(\omega t)^3
\]
\[\quad + \frac{1}{120} A^5 \cos(\omega t)^5 - \frac{\Lambda}{2} \left( 2A\cos(\omega t) \right)
\]
\[\quad + \frac{4}{3} A^3 \cos(\omega t)^3 + \frac{4}{15} A^5 \cos(\omega t)^5 \]

… (32)

Like previous example weighted residuals can be yield as:

\[
\bar{R}_1(t_i) = - \frac{1}{16} A^2 - \frac{1}{24} \Lambda A^4 + \frac{1}{4} \Lambda A^3 - \frac{1}{2} \Lambda A + \frac{1}{384} A^5
\]

… (33)

\[
\bar{R}_2(t_i) = \frac{1}{2} A - \frac{1}{16} A^3 + \frac{1}{384} A^5 - \frac{1}{2} \Lambda A - \frac{1}{2} A\omega^2
\]
\[\quad - \frac{1}{24} \Lambda A^4 + \frac{1}{4} \Lambda A^3 \]

… (34)

Inserting Eqs(33 and 34) into Eq. (11), angular frequency can be yield as:

\[
\omega_{AFF} = \sqrt{\frac{1}{8} \frac{1}{12} \Lambda A^4 + \frac{1}{2} \Lambda A^2 - \Lambda + \frac{1}{192} A^4}
\]

… (35)

So, function of angle can be obtained as:

\[
\theta(t) = A\cos\left[ \left( \begin{array}{c}
- \frac{1}{8} - \frac{1}{12} \Lambda A^4 + \frac{1}{2} \Lambda A^2 \\
\Lambda + \frac{1}{192} A^4
\end{array} \right) t \right] \]

… (36)

Figures 5 and 6 show the comparison between forth-order Runge-Kutta and analytical solutions in two numerical cases with same domains. Especially, Fig. 6 can be illustrated that AFF has more agreement with Runge-Kutta method.
4 Conclusions

In the present paper, Energy Balance Method and Amplitude Frequency Formulation which are two powerful methods, were applied to the motion equations of non-linear oscillators. In the first problem, the Energy Balance Method and Amplitude Frequency Formulation derived the same results but according to Fig. 6 in comparison with the forth order Runge-Kutta method which is powerful numerical method, in the second problem the Amplitude Frequency Formulation indicates more accuracy and shows less error than Energy Balance Method. This implies that, Amplitude Frequency Formulation has less sensitivity to non-linearity rather than the other proposed approach. In this case, the more the non-linearity problem nature has, Amplitude Frequency Formulation has better consistency.

References