Direct search optimization technique for the solution of inverse nonlinear heat conduction problem

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An iterative procedure is used to calculate the transient temperature in a finite slab with temperature dependent thermal conductivity. Closed-form constant property solution is considered as the initial guess to solve the heat conduction equation. The results obtained by this method are in good agreement with the numerical solution. The iterative method for the solution of the nonlinear heat conduction equation has the advantage that temperatures can be directly found at any specified time and location whereas the numerical approach requires the development of temperature profile from right from the initial state. The present paper includes a direct search optimization method for the estimation of the convective heat transfer coefficient from the transient temperature data measured in one-dimensional finite slab with the linear variation of thermal conductivity with temperature. The direct search optimization method does not need calculation of the sensitivity coefficient. The algorithm does not depend on the future-temperature information. The convective heat transfer coefficients are estimated in a typical rocket nozzle using the measured transient temperature at the outer surface. The results computed by the present algorithm are in good agreement with the numerical solution.

Keywords: Inverse problem, Thermal conductivity, Heat conduction, Rocket nozzle

The high temperature range involved and the considerable variation of thermal conductivity with temperature for space application such as nozzle of rocket, thermal protection of the reentry capsule, and simulation of quenching processes require that the variation of conductivity with temperature be considered in the analysis of inverse problem of heat conduction. The purpose of the present paper is to estimate convective heat transfer coefficient from the measured temperature history at the outer surface of the rocket nozzle by taking into consideration a linear variation of thermal conductivity with temperature.

The solution of the inverse heat conduction problem is based on the sequential future information method1,2 and the whole time domain regularization method3,4. The sequential method may become unstable for small time steps. The regularization approach is efficient for small time step but needs large computational time. The sequential future-temperature information method is computationally efficient as compared to the regularization method. The emphasis is given in-gradient bases local search algorithms for estimation of heat flux. Nonlinear inverse problem for the estimation of time-and space dependent is described by Osman and Beck5. One-dimensional inverse problem of high energy laser heating6 of a target is carried out employing conjugate-gradient method. The heat flux on the drilling surface of the drilling tool is estimated in the inverse heat conduction problem using the steepest-descent method by Huang et al.7. The transient convective heat transfer coefficients in the impinging jet arrays are obtained using the color images captured from the liquid crystal8,9 color ply. The heat flux is evaluated by using on isotherms10 of recorded temperature indicators in the coordinate-time plane. A Bayesian inference approach11 is used for the solution of the inverse heat conduction problem, in which the posterior probability density function of the boundary heat flux is computed given temperature measurements within a conducting solid.

Controlled random search algorithm12-14 starts with an initial sets of randomly chosen in the solution space and then carried out an iterative contraction process towards a global minimum by replacing a single point of the population by a better one in each iteration. The controlled random search method uses continuous objective function not necessarily differentiable. The initial set of points is randomly chosen. The direct search optimization technique is

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independent of the initial guess, values of function, sensitive coefficient, future-temperature information and measured errors in the temperature data. The random search method requires range of parameters to be estimated and stopping criterion of iteration. The major differences between the controlled random search method and the direct search optimization technique are related to the generation of modes of the trial points and starting the points and corresponding function values in array. An inverse problem has been carried out using direct search optimization technique to compute discharge coefficient of vent hole in the heat shield compartment of a typical satellite launch vehicle. Measurements are always associated with noisy data. The inverse problem uses simulated input data supplied to inverse solution model along with a simulated measurement noise.

The aim of the present paper is to estimate convective heat transfer coefficient and surface temperature from the measured transient temperature at the outer surface of a typical rocket nozzle using optimization method and taking into account a linear variation of thermal conductivity with temperature in the heat conduction equation.

**Direct Conduction Problem**

A one-dimensional heat conduction equation having convective heat transfer coefficient at one end \( x = 0 \) is heated by hot gas whereas the other end \( x = L \) is insulated as shown in Fig. 1. The initial temperature is known throughout the width of the slab. This is direct heat conduction problem because boundary condition is known and the exact solutions can be easily obtained from Ref. 17. The one-dimensional heat conduction equation and the corresponding boundary conditions in non-dimensional form, for transient heat conduction in a stationary, isotropic, homogeneous solid in Cartesian coordinate system for a finite thickness slab can be written as

\[
\frac{\partial \theta}{\partial t}(X,t) = \frac{\partial}{\partial x} \left[ K(X,t) \frac{\partial \theta}{\partial x}(X,t) \right] \tag{1}
\]

Subjected to boundary conditions

\[
\frac{\partial \theta}{\partial x}(0,t) = 0 \tag{3}
\]

and, the initial condition

\[
\theta(X,0) = 0 \tag{4}
\]

We now consider the constant property solution guess to the forgoing nonlinear problem. The solution procedure is adopted similar to a numerical perturbation technique utilizing the exact solution in conjunction with an iterative scheme. Employing the iterative method, the effect of the variable thermal conductivity on the temperature distribution is taken into consideration by assuming that at any section \( X \) in the region \( 0 \leq X \leq 1 \), the thermal conductivity remains constant over a small interval \( \Delta X \). It is appropriate to mention here that an iterative scheme based on numerical perturbation technique is used in solving the nonlinear heat conduction equation with finite difference discretization method

\[
\theta^{i+1}(X,t) = 1 - 2 \sum_{n=1}^{X} \frac{Bi}{(Bi^2 + \lambda_n^2 + Bi^2) \cos \lambda_n (1 - x)} e^{-\lambda_n^2 (1 + \beta \theta^i (x,t))} \tag{5}
\]

\[
\lambda \tan \lambda = Bi \tag{6}
\]

In the iterative scheme, terms on the right-hand side of Eq. (5) are evaluated from the \( i^{th} \) iteration to obtain \( (i+1)^{th} \) iteration directly from the left-hand side of the equation. It is important to note here that the calculation continues until convergence within the tolerance \( \varepsilon \) is achieved. The \( \lambda_m \) depends on \( Bi \) which is a function of the current value of the thermal conductivity at the surface and are the same for all \( X \) in the region \( L \). \( \theta^i \) and \( \theta^{i+1} \) are known thermocouple temperature in case of the inverse heat conduction problem, therefore, this iteration is not required on
that station, $X$. However, it is required in the calculation of temperature field other than the thermocouple location.

Temperature distribution in a finite slab with $Bi = 1$, at $t = 0.4$ and $1.0$ for $\beta = +0.5$ and $-0.5$ is carried out using the iterative method (ITS). Since no exact solution is available in the case of a finite slab, the present solution is compared with finite difference scheme (FDS). In the numerical solution, this uses two-time level Crank-Nicholson implicit procedure in conjunction with the Taylor’s Forward Projection method. In the numerical solution, this uses the iterative method (ITS). Since no exact solution is available in the case of a finite slab, the present solution is compared with finite difference solution. The inverse heat conduction problem is mathematically ill-posed problem. A thermocouple measurement transient temperature data. The inverse heat conduction problem. A thermocouple at a distance $x_m$ from the boundary of the heated surface as shown in Fig. 1. The measurement temperature data along with the known geometrical detail and thermo-physical properties give rise to the inverse heat conduction problem. A thermocouple may be located between 0 and $L$. The temperature response at an interior point is time lagged and damped with respect to change of heating at the surface. The time interval for computation is taken on-line from the measured transient temperature data. The inverse heat conduction problem is mathematically ill-posed problem. In the estimation of $B_i$, one minimizes

$$F (Bi) = |\theta_c (X,t) - \theta_m (X,t)| \quad \text{... (7)}$$

where $\theta_c$ and $\theta_m$ are, respectively, the calculated and the measured temperatures in the slab.

**Direct Search Optimization Technique**

An optimization method based on direct and systematic search region reduction optimization method is adapted to estimate the unknown convective heat transfer coefficient in a typical rocket nozzle. The most attractive feature of the direct search scheme is the simplicity of computer programming. The pseudo-random algorithm, an effective tool for optimization, does not need computation of derivatives but depends on function $F$ evaluation alone. It works even when the differentiability requirements cannot be ensured in the feasible domain. For imitating the search an estimate of the feasible domain is needed. Therefore, another advantage of the method is that the starting condition is not crucial; any reasonable value will do.

**Algorithm**

The following steps are required to estimate the heat transfer coefficient:

1. Assume the initial value $(h_0)$ and range of $r$ and set the iterative index, $j = 1$.
2. Generate sufficient random numbers between -0.5 and +0.5. Maximum number of iterations (say $j_{max} = 200$).
3. Take a random number $y_k$ from step 2 and assign these to initial values of $h$ and calculate a new value of $h$ in conjunction with the solution of heat conduction equation.
4. Check the function $F$ with each admissible value of $h$.
5. Find the $h$ that minimizes $F$ and stored corresponding value of $(h_0)$, increasing the index from $j$ to $j+1$.
6. Check the maximum number of allowed iterations.
7. Reduce the range by an amount $\varepsilon$ (say $\varepsilon = 0.05$), then
   $$r^j = -(1-\varepsilon)r^j \quad \text{... (9)}$$
8. Go to (ii) and continue.

A flow chart of the direct random search algorithm is given in Fig. 2.
Example

The iterative procedures discussed in the previous section have been employed for estimating the convective heat transfer coefficient and surface temperatures for a typical rocket nozzle divergent of mild steel in conjunction with experimentally measured outer surface temperature data. The nozzle condition and material properties are:

- \( L = 0.02011 \text{ m} \),
- \( T_0 = 300 \text{K} \),
- \( \rho = 7900 \text{ kg/m}^3 \),
- \( C_p = 545 \text{ W.s/kg.K} \),
- \( T_g = 2946.2 \text{ K} \), and running time of motor 16 s.

The values of \( k_0 \) and \( \beta \) used in the present work are 57 W/m.K and – 1.57\(^{\circ}\), respectively. Insulated chromel/alumel thermocouples of 30 gauges are used for measuring the temperature. The inverse heat conduction problem, in turn, utilizes this transient data to determine unknown surface conditions. Calculated results are given in Table 2.

The results of the present analysis are compared with the finite difference solution. The nonlinear heat conduction equation is solved numerically by using two time level Crank-Nicholson implicit method, while a Taylor’s Forward Projection method\(^{20}\) is used to take into consideration the nonlinearities for achieving unconditional computational stability. Twenty space intervals and a time increment of 1 s are used for computational purposes. Initial time step of 6 s is taken for starting the solution. Table 2 shows that results obtained by present analysis are in good agreement with the finite difference solution.

Conclusions

Nonlinear heat conduction equation is solved using previously developed iterative technique. The solution of the nonlinear heat conduction is solved using previously developed iterative scheme. The present analysis includes the temperature dependence of thermal conductivity in the estimation of heat transfer coefficient in conjunction with the direct and systematic search region reduction. Based on known thermocouple temperature, the convective heat transfer coefficient is calculated in a rocket nozzle. The iterative scheme of solving the temperature dependent thermal conductivity in conjunction with the optimization by direct search algorithm has the advantage that temperature found directly at specified time and location whereas the numerical approach needs the development of temperature profile from the initial state.

Nomenclature

- \( Bi = \) Biot number, \( hL/k(\theta) \)
- \( C_p = \) specific heat of material
- \( h = \) convective heat transfer coefficient
- \( K = k_0(1+\beta\theta) \)
- \( k_0 = \) reference thermal conductivity at \( T = T_0 \)
- \( L = \) slab of finite thickness
- \( t = \) dimensionless time, \( \alpha_0\sqrt{L^2} \)
- \( T = \) temperature
- \( T_g = \) combustion gas temperature
- \( T_0 = \) initial temperature
- \( X = \) nondimensional coordinate, \( x/b \)
- \( x = \) axial coordinate

Greek symbols

- \( \alpha_0 = \) reference thermal diffusivity, \( k_0/\rho C_p \)
- \( \beta = \) constant (thermal conductivity coefficient)

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Fig. 2—Flow chart for nonlinear inverse heat conduction problem
\( \rho \) = density of material  
\( \tau \) = time  
\( \theta = \) non-dimensional temperature, \((T - T_0) / (T_g - T_0)\)

**References**