Arbitrary amplitude dust ion acoustic solitary waves and double layers in a kappa distributed electron plasmas

Runmoni Gogoi\textsuperscript{a,*}, Rajkumar Roychoudhury\textsuperscript{b} & Manoranjan Khan\textsuperscript{c}

\textsuperscript{a}Department of Instrumentation Science & Centre for Plasma Studies, Jadavpur University, Kolkata 700 032, India
\textsuperscript{b}Physics and Applied Mathematical Unit, Indian Statistical Institute, Kolkata 700 108, India
\textsuperscript{c}Present address: Assam down town University, Panikhaiti, Guwahati 781 026, Assam, India

E-mail: arunmoni@gmail.com, brajdaju@rediffmail.com c mkhan_ju@yahoo.com

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To investigate dust ion-acoustic solitary waves in an unmagnetized plasma, inertial warm ions, and static dust are considered with electrons following the kappa ($\kappa$) velocity distribution. Using Sagdeev’s pseudo potential method, an exact analytical expression for the Pseudo potential is derived. The range of parameters for the existence of solitary waves and double layers, using the analytical expression of the Sagdeev potential has been found. It is observed that, depending on the values of the plasma parameters like ion temperature $\sigma=T_i/T_e$, kappa spectral index $\kappa$ and the value of the dust grain charge $\mu Z_d$ ($\mu=n_d/n_{d0}$), both rarefactive and compressive solitary waves may exist. Critical values of these parameters, beyond which solitary waves would cease to exist, are obtained for some particular cases. Exact numerical results are obtained for arbitrary amplitude solitons and double layers.

Keywords: Solitary wave, Double layers, Sagdeev potential, $\kappa$-Distribution

1 Introduction

Interaction of dust particles with plasma environment can alter the collective behaviour of the plasma and can give rise to new kind of wave modes and instabilities. Because of the interesting results coming out of the interaction between dust particles and plasmas, it is a very important area of research. Coexistence of plasma with dust occurs due to the fact that major part of the matter in our universe is in plasma state. Dusty plasma is present in interstellar space, interplanetary space, planetary rings (Jupiter, Saturn, Uranus, Neptune), earth’s atmosphere, laboratories etc\textsuperscript{1-3}. In 1954, the existence of dust in the early solar nebula has been advocated by the Nobel Laureate Hannes Alfven\textsuperscript{4}. Micron or submicron dust particles are either positively or negatively charged depending on their surrounding plasma environment. Numerous researchers\textsuperscript{5-9} have studied different behaviour in dusty plasmas for the last two decades or so.

Recently, the kappa distribution\textsuperscript{10} has been of great interest since it is useful for understanding the physical properties of dusty plasma. However, the kappa distribution was first suggested by Vasyliunas\textsuperscript{11} to model space plasmas. The kappa distribution (having the spectral index $\kappa$) is a velocity distribution which has a high energy tail but approaches the Maxwellian when $\kappa \rightarrow \infty$. In the natural space plasma environment, e.g., planetary magnetospheres, astrophysical plasmas and the solar wind, plasma is, generally, observed to possess a non-Maxwellian high energy tail. Applications of the $\kappa$-distribution function\textsuperscript{12} include, for example, an interpretation of observations in the Earth’s foreshock (for $3<\kappa_e<6$) and solar wind models with coronal electrons satisfying $2<\kappa_e<6$. One needs to take the spectral index $\kappa_e>3/2$ for physical reason. Non-Maxwellian suprathermal/superthermal plasmas are found naturally in the magnetosphere of Earth, Mercury, Saturn and Uranus and in the solar wind. Non-linear properties and their stability properties in Maxwellian plasma have been investigated quite extensively for many years by many researchers. However, in many physical situations, both in laboratory and space, velocity distributions are not Maxwellian. Acceleration mechanisms that may lead to
distributions of a power law form are commonly observed both in space and laboratory. That is why it is believed that these mechanisms can be well modelled by a generalized Lorentzian or κ distribution which is useful in modelling both laboratory and space plasmas\textsuperscript{13}. Basu\textsuperscript{14} has investigated the stability of the hydromagnetic waves in kappa distributed plasma, where it was mentioned that typically, space plasmas are observed to possess a spectral index κ in the range 2-6. Recently, measurements of Saturn’s magnetosphere from the Cassini\textsuperscript{15} team have indicated that the electron distribution is a combination of two kappa distributions, the hot (suprathermal) and the bulk (thermal component). They have mentioned that a typical value for the bulk component is κ = 2 throughout most of the magnetosphere, whereas hot component has a highly variable value of κ, lying between 3 and 9.

Hellberg and Mace\textsuperscript{13} observed a generalized plasma dispersion function for a plasma with a kappa-Maxwellian velocity distribution. They have studied the effect of superthermal electrons and ions on ion-acoustic waves propagating at an angle to a magnetic field. Chuang and Hau\textsuperscript{16} have derived general solution of ion acoustic solitons with kappa distributed electrons, which is found to be well represented the particle distribution observed in space plasmas. They have shown the existence of solitary structures with depleted density in non-Maxwellian plasma. Considering kappa distributed electrons and ions, Baluku and Hellberg\textsuperscript{17} studied both small and large amplitude dust acoustic solitary waves in plasmas. They have mentioned that the kappa distribution has only quantitative, not qualitative effect on the existence domains and only negative potential solitons exist. Pacouh and Abbasi\textsuperscript{18} observed non-linear dynamics of DIA solitons in plasma with non-Maxwellian electrons. They showed that width of the solitons is smaller for smaller κ. Application of κ velocity distribution in various kinetic problems of space plasmas has been studied by Fu and Hau\textsuperscript{19} and the references therein.

In our present work, we have considered inertial warm ions, static dust and electrons with kappa distribution to study arbitrary amplitude dust ion acoustic solitary waves in complex plasma. Complex plasma may consist of dust particles, negative ions/multi ions\textsuperscript{20,21} etc. Here we have taken static dust particles and in view of the typical charging time scale, we assume that the dust charge is constant. Thereafter, we adopt an analytical derivation of pseudo potential in terms of potential Φ without any approximation, which is applicable to arbitrary amplitude solitons. Thus, an exact analytical expression for the Sagdeev\textsuperscript{22} potential is obtained and exact numerical solutions for solitary waves are obtained. Both compressive and rarefactive solitary waves are observed.

### 2 Basic Equations

Inertial warm ions, kappa distribution for electrons, and immobile negative charge dust grains in an unmagnetized dusty plasma have been considered. For simplicity, we assume that all the grains have the same charge equal to q_d = -e z_d, with z_d is charge number of dust, and e is the elementary charge.

A generalized three dimensional kappa distribution is given by\textsuperscript{13}:

\[
F_e(v_e) = N_e \left( \frac{\pi \kappa \theta_e^2}{\Gamma(\kappa+1)} \right) \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-1/2)} \left( 1 + \frac{v_e^2}{\kappa \theta_e^2} + 2q_e \phi / m_e \right)^{-\left(\kappa+1\right)}
\]

where \( q_e \) is the species charge of the electrons, \( \Pi \) is the local electrostatic potential, \( v_e \) and \( N_e \) are the electron species velocity and equilibrium number density, respectively. \( \theta_e \) is the generalised thermal speed related to the true thermal speed of the electrons \( v_{th,e} = (2K_B T_e/m_e)^{1/2} \) by \( \theta_e[(\kappa−3/2)/\kappa]^{1/2}v_{th,e} \) with spectral index \( \kappa>3/2 \). \( K_B \) is the Boltzmann constant. The gamma function \( \Gamma \) arises from the normalization of \( F_e(v_e) \) such that:

\[
\int F_e(v_e) d^3v_e = N_e \theta
\]

The family of velocity distribution includes the Maxwell-Boltzmann distribution for \( \kappa \rightarrow \infty \).

Integrating the kappa distribution over velocity space, the number density for electrons can be obtained as:

\[
N_e(\phi) = N_e \left[ 1 - \frac{2q_e \phi}{m_e \kappa \theta_e^2} \right]^{-\left(\kappa-1/2\right)}
\]

Thus, after normalization, the basic equations for one dimensional DIA waves can be written as follows:

For ions

\[
\frac{\partial N_i}{\partial t} + \frac{\partial}{\partial x}(N_i v_i) = 0
\]
\[
\frac{\partial V_i}{\partial t} + V_i \frac{\partial V_i}{\partial x} = -\frac{\partial \Phi}{\partial x} - 3\sigma N_i \frac{\partial N_i}{\partial x} \quad \ldots (4)
\]

For electrons [from Eq. (2)]

\[
N_e = (1 - \frac{\Phi}{k_e - 3/2})^{-\left(k_e - 1/2\right)} \quad \ldots (5)
\]

We close the set of Eqs (1-3) with the Poisson’s equation:

\[
\frac{\partial^2 \Phi}{\partial x^2} = (1 - \mu Z_d)N_e - N_i + \mu Z_d 
\]

Charge neutrality at equilibrium requires \(\mu Z_d = 1 - n_{de}/n_{e0}\)

We have normalized the ion density \(N_i\) by \(n_{i0}\) (equilibrium ion density), electron density \(N_e\) by \(n_{e0}\) (equilibrium electron density) and dust densities \(n_d\) by \(n_{d0}\) (equilibrium dust density), the ion fluid velocity \(V_i\) by \(C_i = (T_e/m_i)^{1/2}\), time \(t\) by inverse of ion plasma frequency \(\omega_{pi}^{-1} = (m_i/4\pi n_{i0}e^2)^{1/2}\), the space variable \(x\) by Debye length \(\lambda_i = (T_e/4\pi n_{i0}e^2)^{1/2}\), the electrostatic potentials \(\Phi\) by \(T_e/e\). Here \(\sigma = T_e/T_i\), \(\mu = n_{d0}/n_{i0}\), \(\mu Z_d\) is the fraction of the negative charge in the plasma which resides on the dust grains.

### 3 Derivation of Sagdeev Potential Equation

For travelling wave solution, we take the transformation \(\xi = x - Mt\) and get the following expressions, from Eqs (3), (4) and (6)

\[
-M \frac{\partial N_i}{\partial \xi} + \frac{\partial}{\partial \xi} \left(N_i V_i\right) = 0 \quad \ldots (7)
\]

\[
-M \frac{\partial V_i}{\partial \xi} + \frac{1}{2} \frac{\partial}{\partial \xi} \left(V_i^2\right) = -\frac{\partial \Phi}{\partial \xi} - \frac{3\sigma}{2} \frac{\partial}{\partial \xi} \left(N_i^2\right) \quad \ldots (8)
\]

\[
\frac{\partial^2 \Phi}{\partial \xi^2} = (1 - \mu Z_d)N_e - N_i + \mu Z_d \quad \ldots (9)
\]

After integration, Eq. (7) gives:

\[
V_i = M - N_i, \quad \text{or} \quad V_i - M = -\frac{M}{N_i} \quad \ldots (10)
\]

From Eq. (8), we get:

\[
2\Phi = M^2 - \frac{M^2}{N_i^2} + 3\sigma - 3\sigma N_i^2 \quad \ldots (11)
\]

Solving Eq. (11), we can easily get \(N_i\) as explicit function of \(\Pi\) and is given by:

\[
N_i = \left\{\frac{M^2 - 2\Phi + 3\sigma - \sqrt{(M^2 - 2\Phi + 3\sigma)^2 - 12\sigma M^2}}{6\sigma}\right\} \quad \ldots (12)
\]

Eq. (9) gives:

\[
\int_0^\xi \left\{1 - \frac{\Phi}{k - 3/2}\right\}^{-k + 3/2} \left[1 - \frac{\Phi}{k - 3/2}\right]^{-1} d\Phi \quad \ldots (13)
\]

From Eq. (5), we obtain:

\[
\int_0^\xi \left[1 - \frac{\Phi}{k - 3/2}\right]^{-k + 3/2} \left[1 - \frac{\Phi}{k - 3/2}\right]^{-1} d\Phi \quad \ldots (14)
\]

Next our aim is to find \(N_i \Phi\) which, after some algebraic calculations, gives us the following result:

\[
\int_0^\xi \left[2(M^2 - 2\Phi + 3\sigma) - \sqrt{(M^2 - 2\Phi + 3\sigma)^2 - 12\sigma M^2}\right] d\Phi \quad \ldots (15)
\]

During integration, we have used the boundary conditions \(N_i \rightarrow 1, V_i, \Phi \rightarrow 0\) as \(\xi \rightarrow \pm \infty\)

Using Eqs (14) and (15) in Eq. (13), we obtain the form:

\[
\int_0^\xi \left[2(M^2 - 2\Phi + 3\sigma) - \sqrt{(M^2 - 2\Phi + 3\sigma)^2 - 12\sigma M^2}\right] d\Phi \quad \ldots (16)
\]
which can be written in the form:

\[
\frac{1}{2} \left( \frac{d\Phi}{d\xi} \right)^2 + \Psi(\Phi) = 0 \tag{17}
\]

with Sagdeev potential as:

\[
\Psi(\Phi) = (1 - \mu_Z) \left\{ 1 - \left( 1 - \frac{\Phi}{k_e - 3/2} \right)^{-(k_e - 1/2)} \right\} - \mu Z_d \Phi + M^2 + \sigma
\]

\[
\frac{1}{3\sqrt{6}\sigma} \sqrt{-\sqrt{(M^2 - 2\Phi + 3\sigma)^2 - 12\sigma M^2}} \times \left\{ 2(M^2 - 2\Phi + 3\sigma) + \sqrt{(M^2 - 2\Phi + 3\sigma)^2 - 12\sigma M^2} \right\} \tag{18}
\]

Eq. (18) is an exact analytical expression for the Sagdeev potential.

It can be seen from Eq. (18) that with limit \( \kappa_e \to \infty \) and \( \sigma \to 0 \), Eq. (18) reduces to:

\[
\Psi(\Phi) = (1 - \mu Z_d) \left\{ 1 - \exp(\Phi) \right\} - \mu Z_d \Phi
\]

\[
+ M^2 - M\sqrt{M^2 - 2\Phi} \tag{19}
\]

This is in agreement with the results obtained by Bharuthram and Shukla\textsuperscript{23} provided we write the symbols as \( 1 - \mu Z_d = N_e \), which implies \( \mu Z_d = 1 - N_e = N_r \).

The analytical solution of Eq. (17) is obtained in the small but finite amplitude limit, where one can neglect terms of \( O(\Phi^5) \) on the right hand side of Eq. (17).

4 Existence Conditions for Solitary Waves and Double Layers

From Eq. (17), it is clear that \( \Psi(\Phi) \) must be negative to get real solution.

**Solitary waves:**

The conditions for solitary waves are:

(i) \( \Psi(\Phi) = 0 \) at \( \Phi = 0 \) and \( \Phi = \Phi_m \);

(ii) \( \frac{\partial \Psi(\Phi)}{\partial \Phi} \bigg|_{\Phi=0} = 0 \) and \( \frac{\partial^2 \Psi(\Phi)}{\partial \Phi^2} \bigg|_{\Phi=0} < 0 \)

(iii) \( \Psi(\Phi) < 0 \) for \( \Phi \) lying between 0 and \( \Phi_m \), i.e. for \( \Phi_m < \Phi < 0 \), rarefactive solitary waves exist and for \( 0 < \Phi < \Phi_m \), compressive solitary waves exist. Here \( \Phi_m \) is the amplitude of the solitary wave.

**Double layers:**

For existence of double layers, the conditions are:

(i) \( \Psi(\Phi) = 0 \) at \( \Phi = 0 \) and \( \Phi = \Phi_m \);

(ii) \( \frac{\partial \Psi(\Phi)}{\partial \Phi} \bigg|_{\Phi=0} = 0 \) and \( \frac{\partial \Psi(\Phi)}{\partial \Phi} \bigg|_{\Phi=\Phi_m} = 0 \)

(iii) \( \frac{\partial^2 \Psi(\Phi)}{\partial \Phi^2} \bigg|_{\Phi=\Phi_m} < 0 \)

Using soliton conditions (i) and (ii), we can get an analytical condition to find a range of the Mach number \( M \) for the existence of solitary waves and double layers as well. It can be easily checked that the conditions \( \Psi(\Phi) = 0 \) and \( \Psi'(\Phi) = 0 \) at \( \Phi = 0 \) are satisfied. The condition \( \Psi''(\Phi) < 0 \) at \( \Phi = 0 \), gives rise to the condition:

\[
\frac{1}{M^2 - \sigma} < \frac{(k_e - 1/2)}{(k_e - 3/2)} (1 - \mu Z_d)
\]

or

\[
M^2 - \sigma > \frac{(k_e - 3/2)}{(1 - \mu Z_d)(k_e - 1/2)} \tag{20}
\]

Again, Eq. (6) can be written in the form:

\[
\frac{d^2 \Phi}{d\xi^2} = -\frac{\partial \Psi(\Phi)}{\partial \Phi}
\]

Using the above relation, setting \( d\Psi/d\Phi = 0 \) for double layers solutions, and using Eqs (6) and (12), we get:

\[
M^2 = \frac{(3\sigma - 2\Phi_m)g^2 - 3\sigma g^4}{1 - g^2} \tag{21}
\]

with \( g = (1 - \mu Z_d)N_e + \mu Z_d \)

and \( N_e \) can be substituted from Eq. (5). Therefore, the range of Mach number \( M \) for different plasma parameters can be obtained from the expression:

\[
M^2 = \left\{ (3\sigma - 2\Phi_m) \left[ (1 - \mu Z_d)(1 - \frac{\Phi_m}{k_e - 3/2})^{-(k_e - 1/2)} + \mu Z_d \right] \right\}^2
\]
where $\Phi_m$ is the amplitude of the double layers [it is the value ($\neq 0$) where both $\Phi$ and $\text{d}\Phi/\text{d}\Phi$ vanishes]. From Eqs (22) and (18), we calculate $\Phi_m$, and then $M$, putting this value of $M$ in $\Psi(\Phi)$.

5 Results and Discussion

Arbitrary amplitude dust ion acoustic solitary waves and double layers in a non-Maxwellian plasma with warm adiabatic ions, static dust and kappa distributed electrons have been investigated. One of the advantages in employing $\kappa$-distribution lies in the fact that the Maxwellian distribution is a special case of the $\kappa$-distribution function in the limit of $\kappa \to \infty$. To observe the existence of solitary wave, we have plotted $\Psi(\Phi)$ against $\Phi$ for different values of the plasma parameters. During our numerical calculations, we have chosen the plasma parameters in such a way that condition given in Eq. (20) is satisfied. Also, the parameters are suitable for laboratory as well as space and astrophysical situations. From the charge neutrality condition $\mu Z_d = 1 - n_d/e n_i$, it is clear that $\mu Z_d < 1$. The temperature ratio, $\sigma = T_i/T_e$ should be less than 1 since $T_e >> T_i$. As mentioned before, for space plasma, kappa parameter $\kappa_e = 2.6$. The Maxwellian and kappa distribution differ substantially in the high energy tail, but the difference become less significant as $\kappa$ increases.

The effect of $\kappa_e$, $\mu Z_d$, $M$, and $\sigma$ on the formation of solitary wave structures has also been studied. FigURE 1 shows the existence of compressive solitary wave for fixed $\mu Z_d = 0.64$, $\sigma = 0.1667$, $M = 1.8$ and different values of $\kappa_e$. It is observed that amplitude decreases as $\kappa_e$ increases. We have varied $\kappa_e$ from 1.6 to 2.2 and noticed that at $\kappa_e = 2.2$, the soliton condition is violated; that is $M^2 - \sigma < (\kappa_e - 3/2) \times \mu Z_d (\kappa_e - 1/2)$, which is clear from Table 1.

In Fig. 2, $\Psi(\Phi)$ against $\Phi$ is plotted for $\kappa_e = 2$, $\sigma = 0.1667$, $M = 1.8$ and different values of $\mu Z_d$. Only rarefactive solitary wave waves exist and amplitudes are found to be decreasing as $\mu Z_d$ increasing. Amplitude decreases in such a way that at some value of $\mu Z_d$, the soliton condition is not satisfied and solitary wave solution ceases to exist. It can be also shown in a similar manner as presented in Table 1. However, for this parameter set, solitary wave solution does not exist when $\mu Z_d \geq 0.75$. Thus, the presence of static dust minimises the amplitude of the solitary wave. Amplitudes of solitary wave increase with the Mach number $M$, as shown in Fig. 3. It has been calculated.
from the soliton condition that for the existence of solitary wave, $M$ must be greater than 1.6. The temperature ratio $\sigma$ also plays important role in the formation of the non-linear structures which can be observed from Fig. 4. Here, we have fixed $\mu Z_d = 0.64$, $M = 1.8$, $\kappa_e = 2$ and varied $\sigma$. We have noticed that amplitudes of solitary wave increase as $\sigma$ increases.

Compressive solitary waves are also found to exist for some parameter sets and for $M$ less than 1. Figure 5 shows the formations of compressive solitary waves for fixed parameters $\mu Z_d = 0.7$, $M = 0.85$, $\sigma = 0.1667$ and different values of $\kappa_e$. Like rarefactive solitary wave amplitudes of the compressive solitary waves also decrease as $\kappa_e$ increases. Amplitude decreases in such a way that beyond $\kappa_e = 1.562$, solitary wave solutions cease to exist. For example, at $\kappa_e = 1.562$, we found that $M^2 - \sigma < (\kappa_e - 3/2) / (1 - \mu Z_d)(\kappa_e - 1/2)$ and hence, violating the existence condition of soliton. The amplitudes of the rarefactive solitary waves are much greater than amplitudes of the compressive solitary waves (Figs 4 and 5). Since compressive solitary waves are weaker they lost their existence faster.

Figure 6 shows the co-existence of both compressive and rarefactive solitary waves for the parameter set $\kappa_e = 1.56$, $\mu Z_d = 0.7308$, $M = 0.865369$, $\sigma = 0.1668$ and $\sigma = 0.1667$. Here it is seen also very clearly that
amplitude of the rarefactive solitary wave is much greater than amplitude of the compressive solitary wave. We can easily check that \( M^2 - \sigma = 0.2485 \) and \( (\kappa_e - 3/2)/(1 - \mu Z_d) (\kappa_e - 1/2) = 0.2363 \) and thus, satisfying the soliton condition. Double layer, a typical electrostatic structure, which shows net potential difference, is responsible for particle acceleration in laboratory as well as auroral plasma region. To investigate the existence of DL, we proceed as mentioned in Sec. 4. Thus, formation of a rarefactive double layer is shown in Fig. 7 to exist for the parameter set \( \mu Z_d = 0.7308, \kappa_e = 1.56, \sigma = 0.1668 \).

6 Conclusions

Both dust & kappa distributions are ubiquitous in space; therefore, result in their interaction with plasma, which motivated us to investigate the present model. Here we have applied the Sagdeev’s Pseudopotential approach which is a very versatile method to observe non-linear activities in space & astrophysical plasmas. The set of equations governing the warm ion dynamics, kappa distribution for electrons & static dust has been reduced to a single equation known as Sagdeev potential equation. Exact analytical expression for the energy integral is obtained and analyzed numerically through which compressive & rarefactive solitary waves & arbitrary amplitude rarefactive double layers are found to exist for different regions of various plasma parameters. Our present theoretical studies could be of interest in laboratory & space plasma, where electrons follow kappa distribution in presence of dust.

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