Sub-Nyquist sampling of high-speed repetitive waveforms using compressed sensing

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This study presents a sub-Nyquist sampling model using compressed sensing (CS) as a new signal processing framework to acquire and reconstruct sparse signals. High-speed periodic signals were sampled using low frequency sampling circuit and reconstructed via CS recovery algorithm, resulting in a high equivalent sampling frequency. This prototype system is able to capture repetitive waveforms at an equivalent sampling rate of 2.5 GHz while sampling at no more than 50 MHz physically.

Keywords: Compressed sensing (CS), Equivalent sampling frequency, Interleaved sampling, Signal reconstruction

Introduction

Shannon theorem lies at the heart of all analog to digital converters (ADC). Improvement in state-of-the-art ADC is not sufficiently fast to catch up with emerging applications in related fields, and many alternative sampling methods have been proposed[1-3]. Interleaved sampling (IS) technique, which is widely used in electronic measurement instrument, extends high-speed capabilities of ADC by an integer factor that equates to the number of ADCs used. In practical application, information rate[4] of a continuous time signal is smaller than its bandwidth presented in signal, and Shannon sampling theorem is sufficient but not necessary. For this family of analog signals, compressed sensing (CS) theory[5,6] ensures that very few random time samples acquired at sub-Nyquist rate will be sufficient to reconstruct analog signal with great precision. Some important theoretical works[7,8] demonstrate feasibility of applying CS to sample spectrally sparse periodic signals, and an implementation of random demodulate sub-Nyquist sampling based on CS was found an effective solution to reduce sampling frequency \( f_s \) without information loss in reconstructed signal. However, signal needs to be modulated by a pseudorandom sequence with Nyquist rate before sampling.

Based on CS theory, this study developed a sub-Nyquist sampling implementation, which enables sub-Nyquist sampling and reconstructing of high-speed periodic signal that is sparse in frequency domain. Experimental waveforms were presented to evaluate proposed implementation.

Experimental Section

Compressed Sensing (CS)

CS states that a sparse or near sparse signal can be recovered from a small salient set of random projections. To make it possible, there are two fundamental premises[5]: sparsity, which pertains to signals of interest; and incoherence, which pertains to sensing modality. An \( N \times 1 \) discrete time signal \( x \) is called \( K \)-sparse \((K << N)\) in some sparsity basis matrix \( \Psi \), which means that there exists a vector \( a \) with \( \|a\|_0 = K \) such that \( x = \Psi a \). Here \( l_0 \) norm \( \|\cdot\|_0 \) counts the number of nonzero entries in vector \( a \). In practice, most of man-made and natural signals are sparse or near sparse, such as Fourier basis. On the contrary, incoherence means that measurement matrix \( \Phi \) has dense representation in the basis \( \Psi \), and \( \Phi \) is independent of \( \Psi \). For \( K \)-sparse signal \( x \), one can find its \( M = O(K \cdot \log(N/K)) \) random projections \( y = \Phi x = \Phi \Psi a \) \((\Phi \) is a matrix with size of \( M \times N \) \((M << N)\), and \( y \) is measurement vector with size of \( M \times 1 \)) and original signal \( x \) can be precisely recovered by solving \( l_0 \) norm optimization problem as

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and it can be an arbitrary value within the interval \([0, T_s]\) 
\(T_s\) is period of sampling clock). Waveform with high
equivalent \(f_s\) is reconstructed via CS recovery algorithm
rather than simply time-interleaved reconstruction method
used in IS technique.

**Signal Reconstruction**

**Measurement Matrix**

In CS, incoherence means measurement matrix (\(\Phi\))
and representation basis (\(\Psi\)) should be as incoherent as
possible. Coherence between \(\Phi\) and \(\Psi\) is given as\(^{10}\)
\[
\mu(\Phi, \Psi) = \max_{1 \leq i, j \leq N, i \neq j} \left| \langle d_i, d_j \rangle \right|
\]  
\(\text{...}(3)\)

where \(d_i\) and \(d_j\) are column vectors of equivalent
dictionary, \(D = \Phi \Psi\). Coherence measures largest
correlation between any two different column vectors of
\(D\), which plays an important role in signal reconstruction.
Coherence is reported\(^{11}\) to be as small as possible in CS.

This study only considered signal sparse in frequency
domain, and \(\Psi\) is Fourier basis\(^{12}\), which is approximated
by Discrete Fourier Transform (DFT) basis, and \(\Psi_{ik} = e^{-j2\pi ik/N} N^{-1/2}, 1 \leq i, k \leq N\). Since \(\Psi\) is fixed, \(\Phi\) directly
affects value of coherence. Most studies so far focus on
matrices with independent and identical distributed
Gaussian and Bernoulli entries, which though have good
properties, it is not easy to be implemented in real world
case. This study introduces \(\Phi\) specifically for proposed
RDS, which is constructed by well-known Shannon
interpolation formula\(^{13}\). According to Shannon sampling
theorem, observation \(y\) and original signal \(x\) satisfy
following formula

\[
\min \|a\|_0 \quad \text{s.t.} \quad y = \Phi x = \Phi \Psi a
\]  
\(\text{...}(1)\)

**Implementation of Sampling Model**

A waveform with high equivalent \(f_s\) was interleaved
from samples acquired using four low-speed ADCs
(Fig. 1). Sampling rate is inversely proportional to the
number of ADCs. In IS, samples of signal \(x(t)\) were
obtained on a periodic and non-uniform grid, a subset of
Nyquist grid. Sequence of samples \(x(nT)\) was taken at
Nyquist rate. Fig. 2 depicts possible architecture of IS.
Let \(M\) be the number of channels and \(f_s\) be the equivalent
\(f_e\). Test signal was sampled using \(M\) low-speed ADCs
that are clocked at \(f_s/M\). Signal can be reconstructed
with no information loss if \(f_e\) is not less than Nyquist
rate. Reconstructed signal can be expressed as
\[
\hat{x}[n] = x(kMT_e + pT_e) , \quad n = 1, 2, \ldots N
\]  
\(\text{...}(2)\)

where \(N\) is length of reconstructed signal, \(T_e = 1/f_e\),
\(k = \text{mod}(n, M)\), and \(p = \left\lfloor n/M \right\rfloor\).

IS presents an architectural possibility of improving
equivalent \(f_s\) of data acquisition system. But for a signal
with sparse spectrum, IS would be redundant. This study
proposes a random delay sampling (RDS) architecture
(Fig. 3) using CS. Frequency of sampling clock can be
reduced to \(f_s/N\) (\(N\) is length of reconstructed waveform,
and \(N > M\), so RDS is more efficient than IS to reduce
\(f_e\)). Different from IS, RDS approach delays test signal
(IS delays sampling clock). Delay time of each signal
channel does not need to be an integral multiple of \(T_e\),
that satisfies $T$

where $1 \leq m \leq M$, $1 \leq n \leq N$, $\Delta t_m$ is time interval as described (Fig. 3), and $T_e$ is equivalent sampling period. From Eq. (4), one can establish relations between non-uniform sampling signal $y$ of size $M$ (number of sampling channels in Fig. 3) and uniform equivalent sampling signal $x$ of size $N$ that is to be recovered. Eq. (4) can be represented as

$$
\begin{bmatrix}
\begin{array}{c}
y(\Delta t_1) \\
y(\Delta t_2) \\
\vdots \\
y(\Delta t_M)
\end{array}
\end{bmatrix} =
\begin{bmatrix}
\Phi_{1,1} & \Phi_{1,2} & \cdots & \Phi_{1,N} \\
\Phi_{2,1} & \Phi_{2,2} & \cdots & \Phi_{2,N} \\
\vdots & \vdots & \ddots & \vdots \\
\Phi_{M,1} & \Phi_{M,2} & \cdots & \Phi_{M,N}
\end{bmatrix}
\begin{bmatrix}
x(T_1) \\
x(T_2) \\
\vdots \\
x(N \cdot T_e)
\end{bmatrix}
$$

$\Leftrightarrow$ $y = \Phi x$ \hspace{1cm} (5)

where $\Phi$ is measurement matrix with element as follows:

$$
\Phi(m,n) = \Phi_{m,n} = \sin c \left( \frac{\Delta t_m}{T_e} - n \right) \hspace{1cm} (6)
$$

In proposed implementation, $M$ is number of sampling channels, and it increases with sparsity level $K$ and reconstructed signal length $N$. In practical sampling, in order to reduce cost, one can take more samples from each sampling channel into the reconstruction process to reduce number of sampling channels. $\Phi$ can be revised as

$$
\Phi(m,n) = \sin c \left( \frac{\Delta t_m + p \cdot T_e}{T_e} - n \right) \hspace{1cm} (7)
$$

where $1 \leq m \leq PM$ ($PM < N$), $i = \text{mod}(m, M)$, $p = \lceil \frac{m}{M} \rceil$ ($p$ takes maximal integer not bigger than $m/M$), and $P$ is number of consecutive samples taken from each sampling channel. Samples from $M$ channels need to be interleaved, and measurement sequences with length of $PM$ can be written as

$$
y[m] = x[p], m = 1, \ldots, PM. \hspace{1cm} (8)
$$

where $x[p]$ is sample of $p^{th}$ acquisition of $i^{th}$ sampling channel, and $p$ and $i$ are defined in Eq. (7).

For fixed $N$ and equivalent $f_s$, there is a tradeoff between $f_s$ and number of acquisitions $P$, and $f_s = Pf_s/N$.

**Signal Recovery Algorithm**

Since $M < N$ [for Eq. (7), $PM < N$], there are many possible solutions of Eq. (1). Original signal $x$ can be reconstructed from $y$ by exploring its sparse expression, that is, among all possible $\hat{\alpha}$ that satisfies $y = \Phi \hat{\Psi} \hat{\alpha}$, seek the sparsest. However, $l_0$ norm regularization problem is a NP-hard problem, and presence of $l_0$ norm makes it computationally difficult to be solved. Generally, there are two classes of methods (convex programming and greedy pursuit algorithms). Although convex programming methods are powerful for signal reconstruction, it may be computationally burdensome. Considering computational complexity and difficulty of realization, in this paper, orthogonal matching pursuit algorithm (OMP)$^4$ is adopted to reconstruct signal.

**Experimental Results**

Experimental waveforms (equivalent $f_s$, 2.5GHz), which were reconstructed from samples acquired using low-speed ADCs, were presented to verify performance of proposed implementation. Experimental setup consisted of a signal generator to generate test signals. There were 16 parallel sampling channels, which were isolated by shielding cases. Signal delay time $\Delta t$ of each channel was randomly selected from $[0, T_e]$. Reference waveforms were captured using a digital storage oscilloscope (DSO). Experiments were carried out in a noiseless environment. Measurement matrix was constructed by Eq. (7), samples were time-aligned by Eq. (8), and signal length was $N = 1000$.

A sinusoidal signal, which is sparse in frequency domain, was first considered. Fig. 4a shows measurement and reconstruction, and 10 samples of each
channel were taken as part in reconstruction process ($P = 10$; and $f_s$, 25 MHz). In this experiment, average difference between reconstructed and reference waveform was 2.1 mV. Fig. 4b describes average difference as a function of number of samples ($P$) of each channel involved in reconstruction. For all values of $P$, average differences were no more than 2.5 mV. Reconstruction of a sine-wave signal (Fig. 5) is disturbed by high-frequency noise (Gaussian noise was programmed in signal generator). In this experiment, 20 samples were taken from each channel to reconstruct signal ($P = 20$; and $f_s$, 50 MHz). This experiment evaluates stability and experimental result indicates that proposed implementation is robust to the noise.

Sparsity is fundamental premise in CS, and performance of reconstruction highly depends upon sparsity of signal. For square-wave signal (Fig. 6), which has infinite number of frequency components, 20 samples were taken from each channel to reconstruct the signal. Due to non-sparse, significant distortion existed in waveform that was reconstructed using CS recovery algorithm. In this experiment, CS recovery method does not improve reconstruction performance significantly, and average difference between reconstructed and reference waveform was 6.62 mV.

**Conclusions**

A prototype of RDS implementation was developed using CS, which enables to sample high-speed repetitive
signals at low sampling rate physically. Measurement matrix in RDS architecture was constructed by Shannon interpolation formula. Prototype was successfully implemented by hardware circuits. Using CS theory, one can reconstruct signal with high equivalent $f_s$ from a small number of samples captured using low-speed ADCs. Using proposed method to sample non-sparse signal, due to its infinite number of frequency components, significant distortion existed in reconstructed waveform. Performance of RDS method is directly tied to sparsity level of test signal, and RDS method can be used as a complement to traditional parallel sampling.

References