Modeling of capacitated single allocation hub location problems with n-hub center

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Hub location problem is NP-hard problem that frequently applies in design of transportation and distribution systems, postal delivery networks, and airline passenger flows. This study presents modeling of capacitated Single Allocation Hub Location Problem (SAHLIP) with two indices in order to get fast model solving.

Keywords: Capacitated, Hub-and-spoke, Hub location problem, Network design

Introduction

Hub and spoke networks (HSNs) are commonly used in industry, particularly in transport, telecommunications and freight, as well as in distributed computing. Parcel network is reported to be based on HSN. Because of rich applications in the real world, studies on various hub location models have attracted much attention. HSNs have application in passenger airlines, express package delivery firms, message delivery networks, trucking industry, telecommunication systems, supply-chain of chain stores such as Wallmart, and many other areas. Hub location problem (HLP) has many varieties according to constraints and decision variables involved such as the way of selecting number of hubs to be located, the way of spokes are assigned to hubs, the existence of capacity limits on hubs, etc. A comprehensive survey on HLPs and their classification is reported.

In Single Allocation Hub Location Problem (SAHLIP), a spoke is allocated to one hub exactly and the number of hubs to be used is not known in advance. Hubs are capacitated SAHLIP (CSAHLIP) with capacity limits on hub or un-capacitated SAHLIP (USAHLIP) involving hubs with unlimited capacities. CSAHLIP application is in postal delivery systems, in which a sorting center (or hub) sorts and consolidates mail arriving from different postal districts and re-route it to destination usually through other centers. Application of USAHLIP is in air transportation networks. SAHLIP is NP-hard problem. CSAHLIP hubs have capacity limits to handle flow between nodes.

Many models and algorithms have been proposed for hub location problem on a network. Excepting O’Kelly model, none of the hub models consider congestion. Some models include capacity limits in terms of traffic. SAHLIP is NP-hard combinatorial optimization problem. Therefore, in recent years, meta-heuristics [Genetic Algorithm (GA), Tabu Search (TS), and Ant Colony Optimization algorithms] have been proposed for SAHLIP. Ernst et al proposed a mixed integer formulation for CSAHLIP and developed two heuristic algorithms for the problem based on simulated annealing and random descent. Matsubayashi studied a cost allocation problem arising from HSN systems. Elhedhli & Hu proposed a model extending current models by taking congestion effects into account. Lin & Chen proposed a generalized HSN in a capacitated and directed network configuration that integrated operations of three common HSNs (pure, stopover and center directs). Klincewica described an algorithm, based on dual ascent and dual adjustment techniques within a branch-and-bound scheme, for un-capacitated hub location problem. He addressed a HSN problem for railroad freight. For USAHLIP, Abdinour-Helm proposed a hybrid approach based on GA and TS. Topcuoglu et al developed a GA-based approach to USAHLIP. Recent models were that by Contreras et al and Naeem. Non-GA heuristics applied to
USAHLP include two hybrid approaches\(^{29}\) that combined SA with Tabu List (TL) to solve USAHLP. This study presents a model with less indices and constraints for CSAHLP.

**Single Allocation Hub Location Problem (SAHLP)**

In SAHLP, a spoke can be assigned to only a single hub. A single allocation HSN is shown in Fig. 1. SAHLP objective is to minimize cost of establishing hubs and cost of transportation. This is subject to constraints that a spoke must be assigned to only a single hub and hub capacities must not be exceeded. USAHLP is a mixed integer formulation and can be found in Naeem’s thesis\(^ {28}\)

\[
\begin{align*}
\min Z &= \sum_{k=1}^{N} f_k z_{kk} + W_{ij} x_{ijkl} + \sum_{k=1}^{N} F_k z_{kk} \quad \cdots (1) \\
\sum_{k=1}^{N} \sum_{m=1}^{N} x_{ijkl} &= 1 \quad \cdots (2) \\
Z_{ik} &\leq z_{ik} \quad \cdots (3) \\
\sum_{i=1}^{N} (W_{ij} x_{ijkl} + W_{jk} x_{ijkl}) &= \left( \sum_{j=1}^{N} W_{ij} + \sum_{j=1}^{N} W_{jk} \right) Z_{ik} \quad \cdots (4) \\
Z_{ik} &\in \{0,1\} \quad \cdots (5) \\
0 &\leq X_{ijkl} \leq 1 \quad \cdots (6)
\end{align*}
\]

where \( n \) is the number of nodes, \( N = \{0,1,2,\ldots,n-1\} \), \( W_{ij} \) is amount of flow between origin \( i \) and destination \( j \), \( x \) is collection cost (from origin spoke to hub), \( b \) is distribution cost (from hub to destination spoke), \( d_{ik} \) represents the distance between nodes \( i \) and hub \( k \), \( d_{ij} \) is distance between hub \( l \) and node \( j \), \( X_{ijkl} \) is decision variable that represents fraction of traffic between origin node \( i \) to destination node \( j \) through hubs \( k \) and \( l \), \( F_k \) is cost of establishing node \( i \) as hub, \( Z_{ij} \) is 1 if node \( i \) is assigned to hub \( j \), otherwise it is 0, and \( Z_{kk} \) is 1 if node \( k \) is also a hub, otherwise it is 0.

Constraint (2) ensures that all traffic between an origin-destination pair has been routed via hub subnetwork. Constraint (3) prevents non-hub nodes from being allocated to other non-hub nodes while Constraint (4) restricts commodity flow through each hub. For some hub-spoke networks e.g., a mail delivery system, problem may not be symmetric as \( W_{ij} \neq W_{ji} \). In case \( W_{ij} > 0 \), a node may route commodities to itself. In this model, both symmetric and non-symmetric flows are employed.

In another mathematical formulation, Contreras et al.\(^ {27}\) model for SAHLP consists of selecting a set of hubs to be established and an allocation pattern that fully assigns each node to one of the chosen hubs, that does not violate capacity constraint of hubs, of minimum total cost.

\[
\begin{align*}
\min Z &= \sum_{k=1}^{N} f_k z_{kk} + W_{ij} x_{ijkl} + \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{m=1}^{N} F_{jkm} x_{ijkl} \\
\sum_{k=1}^{N} \sum_{m=1}^{N} x_{ijkl} &= 1 \quad \cdots (8) \\
Z_{ik} &\leq z_{ik} \quad \cdots (9) \\
\sum_{m=1}^{N} x_{ijkm} &= z_{ik} \quad \cdots (10) \\
\sum_{k=1}^{N} x_{ijkm} &= z_{jm} \quad \cdots (11) \\
\sum_{i=1}^{N} \sum_{j=1}^{N} W_{ij} z_{ik} &\leq b_k z_{kk} \quad \cdots (12) \\
\sum_{k=1}^{N} b_k z_{kk} &\leq D \quad \cdots (13)
\end{align*}
\]
\[ X_{ijkm} \leq 0 \quad \ldots (14) \]
\[ Z_k \in \{0,1\} \quad \ldots (15) \]

Constraint (8) ensures that for each pair of nodes there is one single path connecting them, whereas constraint (9) imposes that no customer is assigned to a node that is not a hub. Constraint (10) states that if node \( i \) is assigned to hub \( k \) then all flow from node \( i \) to any other node \( j \) must go through some other hub \( m \). Constraint (11) has a similar interpretation relative to flow arriving to a node \( j \) assigned to hub \( m \) from some node \( i \). Constraint (8) together with constraints (10) and (11) ensures that every node is assigned to one single hub. In addition, given that \( z \) variables are binary, they guarantee integrality of \( x \) variables. Constraint (12) ensures that overall incoming flow of nodes assigned to a hub does not exceed its capacity. Constraint (13) is aggregated demand constraint, which is redundant in model, since it can be derived by adding up all constraints (12), and taking into account equalities (8) and (10).

Model similar to this model but without aggregated demand constraint (13) was proposed by Campbell\(^{14}\). Computational experiments\(^{12}\) showed that formulation that used variables with four indices lead to tighter LP bounds than the formulation that used variables with three indices, at a considerable increase on required CPU times. Contreras et al\(^{27}\) proposed a Lagrangean Relaxation associated with constraints (10) and (11). In particular, weighting constraints (10) and (11) in a Lagrangean fashion, with multipliers vectors \( u \) and \( v \) of appropriate dimensions, obtain a Lagrangean function which, after some algebra can be expressed as:

\[ L(u, v) = L_z (u, v) + L_x (u, v). \]

**Proposal Model**

Proposed model with two indices solves SAHLPs faster than model with four indices. Additionally, in UCSAHLP with 10 nodes, proposed model will have 67 constraints but Naeem’s model\(^{28}\) and Contreras’s model\(^{27}\) will have 220 and 330 constraints. Therefore, proposed model will be able to solve problems faster than models of Naeem and Contreras.

**Data**

In proposed model, data used was as follows: \( n \) is number of nodes, \( x_{ij} \) is distribution cost (from hub to destination spoke), \( d_{ij} \) represents distance between nodes \( i \) and \( j \), \( F_i \) is cost of establishing node \( i \) as hub and \( C_i \) is capacity of hub \( i \).

**Decision Variables**

In proposed model, decision variables used were as follows: \( w_{ij} \) is amount of flow between hub \( i \) and destination \( j \); \( Z_{ij} \) is 1 if node \( j \) is assigned to hub \( i \), otherwise it is 0; \( Z_{ii} \) is 1 if node \( i \) is also a hub, otherwise it is 0; and \( h \) is number of hubs.

**Uncapacitated Model**

\[ \min Z = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{2} x_{ij} (w_{ij} Z_{ij} d_{ij}) + \sum_{i=1}^{N} F_i Z_{ii} \]

\[ \ldots (16) \]
\[ \sum_{i=1}^{n} Z_{ii} = h \] ... (17)

\[ \sum_{i=1}^{n} \sum_{j=i+1}^{n} Z_{ij} = n - h \] ... (18)

\[ Z_{ij} + Z_{ji} + 2[Z_{ij} + Z_{ji}] \quad \forall i, j = \{i+1, ..., n\} \] ... (19)

\[ z_{ii} \sum_{j=1}^{n} Z_{ij} \geq 2Z_{ii} \quad \forall i \] ... (20)

\[ (1 - z_{ii}) \sum_{j=1}^{n} Z_{ij} \leq 1 \quad \forall i \] ... (21)

\[ w_{ij} \geq 1 \quad \forall i, j \] ... (22)

\[ \sum_{j=1}^{n} Z_{ij} w_{ij} = c_{i} Z_{ii} \quad \forall i \] ... (29)

\[ \sum_{j=1}^{n} Z_{ij} w_{ij} = c_{i} Z_{ii} \quad \forall i \] ... (30)

\[ w_{ij} \geq 1 \quad \forall i, j \] ... (31)

Capacitated model is similar to un-capacitated model but two constraints have been added in this model. Constraints (29) and (30) determine capacity for each hub. In general, there are 14n variables, with 7n binary integer variables and 2 + 4n constraints in proposed model. Thus proposed model is effective with less indices and constraints in solving hub and spoke problems.

Conclusions
This paper presented a model with less indices and constraints for CSAHLP. Comparison results of three models (Naeem, Contreras and Proposed model) represented that proposed model was faster than others. Proposed model solves instances of CSAHLP up to 50 nodes. Considered small and medium instances have been solved exactly for UCSAHLHP in an acceptable time.

References


27 Contreras I, Diaz J & Fernandez E, Branch-and-Price for Large-Scale Capacitated Hub Location Problems with Single Assignment, *OR Spectrum* (Politecnica University, Spain) 2009.
