ON-inclusion of mathematics in the list of the Nobel Prizes has been intriguing many since its very inception. People wonder how Alfred forgot about the mother of physical sciences while conceiving the idea of recognizing the best scientific talents for their contributions to the welfare of mankind. However, now prizes instituted for mathematics equal the Nobel in recognition. Perhaps the foremost are the Millennium Prizes of the Clay Mathematics Institute (CMI). Recently, the eccentric Russian mathematics genius Dr. Grigory Perelman has been nominated for this honour for solving the 106-year-old Poincare Conjecture.

The Millennium Prizes were conceived to honour outstanding mathematicians for solving some of the most difficult problems in mathematics, which eluded solution at the turn of the second millennium. Seven such problems were identified by the Clay Mathematics Institute (CMI) in 2000, which included the Poincare conjecture. These prizes are not meant to be awarded regularly every year like the Abel Prize or the Nobel Prizes. Only when some one solves one of the identified problems, he or she can be nominated for the honour. Therefore, the maximum number of the Millennium Prizes that can be awarded is limited to seven only. That’s the reason why it took almost a decade to find a befitting nominee for this award.

Poincare Conjecture

In 1904, the French Mathematician Henri Poincare posed an epoch making question in one of his papers, which asked, “If a three-dimensional shape is simply connected, is it homeomorphic to the three dimensional sphere?” The statement can be explained by considering the analogous two-dimensional situation.

Let us think of a rubber band stretched around the spherical surface of an apple. It is easily seen that it can be shrunk to a point by moving it slowly, without tearing it and without allowing it to leave the surface. On the other hand, let us replace the apple by a doughnut, which is a torus. The band cannot be shrunk in anyway to a point without tearing it or the doughnut. It can be stated mathematically that the apple is “simply connected”, whereas, the doughnut is not.

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In order to characterize the spherical surface of the apple, we can imagine a disc lying in a two-dimensional plane with its boundary lifted up and tied to a single point. In other words, it can be said that all the points are identified to a single point. It can be done if the two-dimensional disc is lying in a three-dimensional space i.e. we have a two-dimensional sphere in a three-dimensional space. Therefore, Poincare asked, if a two-dimensional sphere is characterized by the property of simple connectivity, a similar characteristic is valid for all closed three-dimensional objects embedded in a four-dimensional space, which are like three-dimensional spheres.

Although analogous results were found to be true at higher dimensions, the case of three-dimensional sphere had been proving to be the hardest. That is why, the question had been viewed as extraordinarily difficult and for long mathematicians were struggling to find an answer to it. Recently, Grigori Perelman succeeded in finding an answer.

Towards the Resolution

In fact, over the years, many outstanding mathematicians, including Poincare himself pondered over the problem. Of course, these led to the creation of new vistas in mathematics, but the conjecture still remained unsolved. For example, in 1961 Stephen Smale of the University of California, Berkeley proved that the analogue of the Poincare conjecture was true for spheres of five or more dimensions. However, it did not answer the original question of Poincare. On the other hand, it suggested that the theory of dimensions three and four was unlike the theory of spheres in higher dimensions. Then, a decade later, Michael Freeman of the University of California, San Diego announced a proof of the Poincare conjecture in dimension four, which could not be extended to dimension three. However, for their work Smale and Freeman were awarded the prestigious Fields Medal in 1966 and 1986 respectively.

Thereafter, three major developments took place in this field, which played crucial roles in guiding Perelman to reach the solution of the conjecture. These were, (i) William Thurston’s geometrization conjecture, (ii) Aleksandrov limits of (geometric) spaces, and (iii) Ricci flow equation.

According to the geometrization conjecture, three-dimensional shapes could be broken down into pieces governed by one of the eight geometries – a similar way as a molecule can be broken down into simpler atoms.

The ideas of limits of space deals with how one can take limits of geometric shapes, just as we take limits in calculus. To understand this, think of Zeno’s paradox: First you walk half the distance from where you are standing, towards the wall of a room. Then you walk half of the remaining distance and go on repeating it. Every time you get closer to the wall, which is your “Limiting position”, but you never reach it in a finite number of steps.

Again let us imagine a Y-shaped rubber tube collapsing with time. As time increases, its diameter gets smaller and smaller. Ultimately, as it becomes zero, the tube becomes an infinitely thin wire. The point where the arms of Y meet is different from others. It is called the “singular point” or “singularity” of this space. The kinds of shapes that occur as limits are called Aleksandrov limits of spaces, named after the Russian mathematician, who initiated and developed the theories.

The development was connected with differential equations involving the rates of change in the two unknown quantities of the equation. Those were applied to geometric and topological problems as well as to physical ones. The differential equation that played a key role in solving Poincare conjecture is the Ricci flow equation. It was discovered twice – once for its application in geometry (mathematics) in 1982 and again in quantum field theory (physics).

Ricci Flow Equation

In geometric context, the Ricci flow equation was the analogue of Fourier’s heat equation. Just as the latter disperses temperature, the former disperses curvature. Therefore, even if a shape is irregular and distorted, Ricci flow can remove these anomalies, resulting in a very regular shape, whose topological nature is evident.

Richard Hamilton, in 1982, showed that for positively curved, simply connected shapes of dimension three, the Ricci flow transformed the shape into one that was even more like the round three-sphere, which in the long run became almost indistinguishable from this perfect, ideal shape. On the other hand, when the curvature was not strictly positive, solutions of the Ricci flow equation behaved in a more complicated manner, as it was nonlinear. As a result, certain parts of the shape may be evolved towards a smoother and more regular state, while the other parts might develop singularities. Although this behavior posed serious difficulties, it also held promise. It was conceivable from it that the formation of singularities could reveal Thurston’s decomposition of a shape into its constituent geometric atoms.

In 1999, Richard Hamilton published a paper showing that in a Ricci flow, the curvature was pushed towards the positive near a singularity. In it he made use of the collapsing theory. Again the Hamilton-Harnack inequality also played an important role in Perelman’s proof. It was generalized to positive Ricci flow for positive solutions of Fourier’s heat equation.

Although Hamilton established Ricci flow equation as a tool to resolve both conjectures and geometric problems, obstacles still remained to a proof of Poincare conjecture. One of those was the lack of understanding of the formation of singularities in Ricci flow, akin to the formation of blackholes in the evolution of the cosmos. At last, in 2002 and 2003 Grigory Perelman found a solution not
only to the Poincare conjecture but also to Thurston’s geometrization conjecture.

**Perelman’s Proof**

Perelman’s method of proof was based on the theory of Ricci flow. To its application in topology, he brought great technical virtuosity as well as new ideas. One of those was combination of collapsing theory in Riemannian geometry with Ricci flow. It gave an understanding of the parts of the shape that were collapsing on to a lower-dimensional space.

Similarly, another was the introduction of entropy. While in the classical theory of heat exchange, it measures disorder in atomic level, here it means disorder in global geometry of space. Perelman’s entropy, like the thermodynamic entropy, is increasing with time and there is no turning back. Using his entropy function and a related version, Perelman was able to understand the nature of singularities that formed under Ricci flow. There were just a few kinds and one could write down simple models of their formation.

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Then it became clear how to cut out the parts of the shape near them as to continue the Ricci flow past the times, at which they would otherwise form. From these results, Perelman showed that the formation of singularities could not run into Zeno’s wall; for example, imagine a singularity that occurs after one second, then after half a second more, then after a quarter second more and so on. If this were to occur, the “wall”, which one would reach two seconds after departure, would correspond to a time at which Ricci flow would cease to hold. The proof would be unattainable. However, with this new mathematics in hand, it was attainable.

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