Multivariable optimization of two-stage thermoelectric refrigerator driven by two-stage thermoelectric generator with external heat transfer

Fankai Meng, Lingen Chen* & Fengrui Sun
Postgraduate School, Naval University of Engineering, Wuhan 430033, P. R. China
*E-mail: lgchenna@yahoo.com; lingenchen@hotmail.com
Received 11 February 2010; accepted 20 August 2010

The finite time thermodynamic model of a combined thermoelectric device, two-stage thermoelectric refrigerator driven by two-stage thermoelectric generator with heat transfer irreversibility is built by combining non-equilibrium thermodynamic theory with finite time thermodynamics theory. The solution procedure of optimal performance and the corresponding optimal variables are given and the cooling load and COP of the combined thermoelectric device are optimized. The control equation of the combined thermoelectric system is obtained. For fixed total number of thermoelectric elements and fixed total external thermal conductance of the device, the allocations of thermoelectric elements and thermal conductance are optimized for maximum cooling load and COP, respectively. The effects of the heat source temperature of the two-stage thermoelectric generator and the heat source (cooling space) temperature of the two-stage thermoelectric refrigerator on the optimal performance and optimal variables selection are analyzed by detailed numerical examples. The numerical simulation results show that the optimization is necessary and effective. By optimization, the performance is improved. When the two-stage thermoelectric generator works at a difference of 150 K, a cooling temperature difference of about 60 K could be reached. When the two-stage thermoelectric refrigerator works at a difference of 20 K, a COP of about 0.10 could be reached. These are considerable for application in a wide-scale for specific purposes.

Keywords: Two-stage, Thermoelectric generator, Thermoelectric refrigerator, Heat transfer, Finite time thermodynamics, Performance optimization

1 Introduction

Semiconductor thermoelectric power generation, based on the Seebeck effect¹, and semiconductor thermoelectric cooling, based on the Peltier effect², have very interesting capabilities³-⁵ compared to conventional power generation and cooling systems. The absence of moving components results in an increase of reliability, a reduction of maintenance, and an increase of system life; the modularity allows for application in a wide-scale range without significant losses in performance; the absence of a working fluid avoids environmental dangerous leakage; and the noise reduction appears also to be an important feature. Thermoelectric generator and refrigerator have been used in military, aerospace, instrument, and industrial or commercial products, as a power generation and cooling devices for specific purposes. Many researchers are concerned about the physical properties of thermoelectric material⁶ and the manufacturing technique of thermoelectric modules⁷-¹³. In addition to the improvement of the thermoelectric material and module, the system analysis and optimization of thermoelectric generator and refrigerator are equally important in designing high-performance thermoelectric generators and refrigerators³-⁵.

In general, conventional non-equilibrium thermodynamics¹⁴-¹⁶ is used to analyze the performance of single-stage one or multiple-element thermoelectric generators¹⁷-¹⁹ and refrigerators²⁰-²³. Due to the performance limits of thermoelectric material, thermoelectric generators and refrigerators of two stages or more should be applied to improve the level of thermodynamic performance. The performance analysis and optimization of two-stage thermoelectric generators²⁴ and refrigerators²⁵-²⁶ were also performed. All of those were performed by using conventional non-equilibrium thermodynamics without considering the external losses. The theory of finite-time thermodynamics or entropy generation minimization²⁷-³⁶ is a powerful tool for performance analysis and optimization of practice thermodynamic processes and devices. Performances of two-stage thermoelectric generators²⁷ and refrigerators²⁸,²⁹ with external heat transfer are analyzed using the combination of finite time thermodynamics and non-equilibrium thermodynamics.

All objects of these researches were dependent thermoelectric devices, that is, they need a direct-current power source providing direct current to refrigerator or heat up. However, for some special
systems, such as submarines, cars, and special electric circuit, the heat rejected from the thermal machine may drive a thermoelectric refrigerator through the use of a thermoelectric generator so that the thermoelectric refrigeration does not need a special power source. Such a new refrigeration system is directly composed of a thermoelectric generator and a thermoelectric refrigerators. It is different from the traditional thermoelectric systems which merely consist of thermoelectric generators or refrigerators, and deserves to be investigated both from the point of view of theoretical design, as well as practical application.\(^{40-45}\) Chen et al.\(^{40}\) and Khattab et al.\(^{41}\) built a model of this kind of combined system, single-stage thermoelectric refrigerator driven by single-stage thermoelectric generator, and analyzed the performance of the device. Chen and Meng et al.\(^{42-45}\) built a model of a single-stage thermoelectric heat pump driven by a single-stage thermoelectric generator,\(^ {42,43}\) a model of two-stage thermoelectric refrigerator driven by a two-stage thermoelectric generator\(^ {44}\) and a model of two-stage thermoelectric heat pump driven by a two-stage thermoelectric generator.\(^ {45}\) The analytical formulae for the stable working electrical current are derived. The effects of the hot junction of the thermoelectric generator and cold (hot) junction temperature of the thermoelectric refrigerator (heat pump) on the optimal performance are analyzed by using non-equilibrium thermodynamic theory.

The work in Ref. (44) was based on non-equilibrium thermodynamics, i.e. the external heat transfers among the hot and cold junctions of the two-stage thermoelectric generator and the two-stage thermoelectric refrigerator and their respective reservoirs were not considered. Thermoelectric devices cannot be used independently. They should be connected with heat exchangers to dissipate heat.\(^ {3,5}\) Much work has shown that the heat transfer irreversibility between the device and its external reservoirs affects the performance of thermodynamic processes.\(^ {27-36}\) Strongly on the basis of the exo-reversible model of two-stage thermoelectric refrigerator driven by a two-stage thermoelectric generator without external irreversibility built in Ref. (44), the finite time thermodynamic model with external irreversibility of the combined thermoelectric device is built by combining non-equilibrium thermodynamic theory with finite time thermodynamics theory in this paper. As is well known, more thermoelectric elements and more external thermal conductance mean larger generated power (cooling load) but bulkier structures and higher cost of the device. It is an interesting and important problem that how to allocate fixed total the thermoelectric elements and external thermal conductance among the two-stage thermoelectric refrigerator and the two-stage thermoelectric generator for maximum cooling load or COP. Besides, how the two-stage thermoelectric generator heat source temperature and the two-stage thermoelectric refrigerator cooling space temperature affect the optimal allocation and the corresponding performance is another interesting problem. These are the main purpose of this research.

2 Finite Time Thermodynamic Model and Fundamental Relations

A schematic diagram of a two-stage combined thermoelectric device, i.e. a two-stage thermoelectric refrigerator driven by a two-stage thermoelectric generator is shown in Fig. 1. The device consists of a two-stage thermoelectric generator and a two-stage thermoelectric refrigerator in series. The direct-current power source of the two-stage thermoelectric refrigerator is the electrical current generated by the two-stage thermoelectric generator.

The irreversible two-stage thermoelectric generator consists of a top stage with \(m_h\) pairs of thermoelectric elements and a bottom stage with \(m_c\) pairs of thermoelectric elements. The total number of thermoelectric element pairs of the two-stage thermoelectric generator is \(m = m_h + m_c\). Each element is composed of a \(p\)-type and an \(n\)-type semiconductor legs. The thermoelectric generating element is assumed to be insulated, both electrically and thermally, from its surroundings, except at the junction-reservoir contacts and the junction between the two-stages. The internal irreversibility is caused by Joule loss and heat conduction loss through the semiconductor between the hot and cold junctions. The Joule loss generates an internal heat \(I^2R\) , where \(R\) is the total internal electrical resistance of the semiconductor couple and \(I\) is the working electrical current generating from the semiconductor couple. The conduction heat losses are \(K(T_{h1} - T_{c1})\) for the top stage and \(K(T_{h2} - T_{c2})\) for the bottom stage, respectively, where \(K\) is the thermal conductance of the semiconductor couple, \(T_{h1}\) is the hot junction temperature, \(T_{c1}\) is the cold junction temperature, and \(T_{h2}\) is the temperature of the junction between the two stages. The rates of heat flow of the two-stage thermoelectric generator are \(Q_{h1}\), \(Q_{c1}\) and \(Q_{c2}\).
The irreversible two-stage thermoelectric refrigerator consists of a top stage with \( m_1 \) pairs of thermoelectric elements and a bottom stage with \( m_2 \) pairs of thermoelectric elements. The total number of thermoelectric element pairs of the two-stage thermoelectric refrigerator is \( m_1 + m_2 \). The structure of the two-stage thermoelectric refrigerator is similar to the two-stage thermoelectric generator. The conduction heat losses are \( K(T - T_0) \) for the top stage and \( K(T - T_0) \) for the bottom stage, respectively, where \( T \) is the hot junction temperature, \( T_0 \) is the cold junction temperature, and \( T_0 \) is the temperature of the junction between the two stages. The rates of heat flow of the two-stage thermoelectric refrigerator are \( Q_{h1}, Q_{h2}, Q_{l1}, \) and \( Q_{l2} \).

Assume that the four heat exchangers among the hot and cold junctions of the two-stage thermoelectric refrigerator, top and bottom stage thermoelectric generator and their respective reservoirs are counter-flow, and the thermal conductance (product of heat transfer coefficient and heat transfer surface area) of the heat exchangers are \( K_{h1}, K_{l1}, K_{h2}, \) and \( K_{l2} \), respectively, where \( K_i = k_iF_i \), where \( k_i \) is the heat transfer coefficient of heat exchanger, and \( F_i \) is the heat transfer surface area of the heat exchanger, respectively.

The total number of thermoelectric elements of the irreversible combined thermoelectric device is finite, \( M = m_1 + m_2 + m_3 \). The total thermal conductance of the four heat exchangers of the irreversible combined thermoelectric device is also finite, \( K_T = K_{h1} + K_{l1} + K_{h2} + K_{l2} \).

It is assumed that that the heat transfers among the hot and cold junctions of the two-stage thermoelectric generator and the two-stage thermoelectric refrigerator and their respective reservoirs obey Newton’s heat transfer law, as did finite time thermodynamics for various thermodynamic processes and devices. According to the theory of non-equilibrium thermodynamics, for the two-stage thermoelectric generator, one has:

\[
Q_{h1} = K_{h1}(T_{h1} - T_{l1}) \quad \ldots(1)
\]

\[
Q_{h2} = m_1[\alpha RT_0 + K(T_{h2} - T_{l2}) - 0.5I^2 R] \quad \ldots(2)
\]

\[
Q_{l1} = m_1[\alpha RT_0 + K(T_{l1} - T_{i1}) + 0.5I^2 R] \quad \ldots(3)
\]

\[
Q_{l2} = m_1[\alpha RT_0 + K(T_{l2} - T_{i2}) - 0.5I^2 R] \quad \ldots(4)
\]

\[
Q_{l3} = m_1[\alpha RT_0 + K(T_{l3} - T_{i3}) + 0.5I^2 R] \quad \ldots(5)
\]

\[
Q_{l4} = K_{l1}(T_{i1} - T_{l1}) \quad \ldots(6)
\]

For the two-stage thermoelectric refrigerator, one has:

\[
Q_{h2} = K_{h2}(T_{h2} - T_{h1}) \quad \ldots(7)
\]

\[
Q_{h2} = m_2[\alpha RT_0 - K(T_{h2} - T_{h2}) + 0.5I^2 R] \quad \ldots(8)
\]

\[
Q_{l2} = m_2[\alpha RT_0 - K(T_{l2} - T_{l2}) - 0.5I^2 R] \quad \ldots(9)
\]
where \( \alpha = \alpha_p - \alpha_n \), \( \alpha_p \) and \( \alpha_n \) are the Seebeck coefficients of the \( p \)- and \( n \)-type semiconductor legs for each two-stage thermoelectric generator (refrigerator) thermoelectric element.

Combining Eqs (1) with (2), gives the relation between \( T_{m1} \) and \( T_{m2} \):

\[
T_{m2} = \frac{Q_{m2}}{m_{n2}[\alpha IT_{m2} - K(T_{n2} - T_{c2}) + 0.5I^2R]} \quad \ldots(10)
\]

Substituting Eq. (15) into Eqs (13) and (14), one can obtain the temperatures of the hot and cold junctions of the two-stage thermoelectric generator.

Combining Eq. (7) with (8), gives the relation between \( T_{h2} \) and \( T_{m2} \):

\[
T_{h2} = \frac{-(K_{h2}T_{h2} + m_{n2}KT_{m2} + 0.5m_{n2}I^2R)}{m_{n2}\alpha I - m_{n2}K - K_{h2}} \quad \ldots(18)
\]

Combining Eqs (11) with (12) gives the relation between \( T_{e2} \) and \( T_{m2} \):

\[
T_{e2} = \frac{0.5m_{n2}RI^2 + m_{n2}KT_{m2} + K_{e2}T_{e2}}{m_{n2}\alpha I + m_{n2}K + K_{e2}} \quad \ldots(19)
\]

Substituting Eqs (18) and (19) into Eqs (9) and (10), one can obtain the temperature of the junction between the two stages of the two-stage thermoelectric generator:

\[
T_{m2} = \frac{N_{r_{m2}}}{D_{r_{e2}}} \quad \ldots(20)
\]

where

\[
N_{r_{m2}} = \frac{(Rm_{n2}\alpha^2m_{e2} + Rm_{n2}\alpha^2m_{n2})I^4 + [-\alpha R(-m_{n2}m_{n2}K - m_{n2}m_{n2}K + K_{h2}m_{n2} + m_{n2}K_{h2}m_{n2})]I^3 + (R(2m_{n2}m_{n2} + 2Km_{n2}K_{h2}m_{n2} + m_{n2}K_{h2}K_{h2}m_{n2}m_{n2} + K_{h2}m_{n2}m_{n2}))I^2 + (2m_{n2}K_{h2}m_{n2}m_{n2} + m_{n2}K_{h2}K_{h2}m_{n2}m_{n2} + K_{h2}m_{n2}m_{n2})I + 2(m_{n2}K_{h2}K_{h2}m_{n2} + m_{n2}K_{h2}K_{h2}m_{n2} + m_{n2}K_{h2}K_{h2}m_{n2})}{(2m_{n2}K_{h2}m_{n2}m_{n2} + m_{n2}K_{h2}K_{h2}m_{n2}m_{n2} + K_{h2}m_{n2}m_{n2})I^3 + (2m_{n2}m_{n2}K_{h2}m_{n2}m_{n2} + 2m_{n2}K_{h2}m_{n2}m_{n2} + m_{n2}K_{h2}m_{n2}m_{n2} + K_{h2}m_{n2}m_{n2})I^2 + (2m_{n2}K_{h2}m_{n2}m_{n2} + m_{n2}K_{h2}K_{h2}m_{n2}m_{n2} + K_{h2}m_{n2}m_{n2})I + 2(m_{n2}K_{h2}m_{n2}m_{n2} + m_{n2}K_{h2}K_{h2}m_{n2}m_{n2} + m_{n2}K_{h2}K_{h2}m_{n2})}
\]

\[
D_{r_{e2}} = \frac{(2\alpha^2m_{n2}m_{n2} - \alpha^2m_{n2}m_{n2})I^4 + (2\alpha^2(m_{n2}m_{n2}K + m_{n2}K_{h2}m_{n2} - m_{n2}K_{h2}m_{n2} - m_{n2}K_{h2}m_{n2})I^3 + (2\alpha^2(-m_{n2}m_{n2}K + m_{n2}K_{h2}m_{n2} + m_{n2}K_{h2}m_{n2} + K_{h2}m_{n2}m_{n2})I^2 + (2\alpha^2(-m_{n2}m_{n2}K + m_{n2}K_{h2}m_{n2} + m_{n2}K_{h2}m_{n2} + K_{h2}m_{n2}m_{n2})I + 2(2\alpha^2(m_{n2}m_{n2}K + m_{n2}K_{h2}m_{n2} - m_{n2}K_{h2}m_{n2} - m_{n2}K_{h2}m_{n2}))}{(2\alpha^2(m_{n2}m_{n2}K + m_{n2}K_{h2}m_{n2} - m_{n2}K_{h2}m_{n2} - m_{n2}K_{h2}m_{n2})I^3 + (2\alpha^2(-m_{n2}m_{n2}K + m_{n2}K_{h2}m_{n2} + m_{n2}K_{h2}m_{n2} + K_{h2}m_{n2}m_{n2})I^2 + (2\alpha^2(-m_{n2}m_{n2}K + m_{n2}K_{h2}m_{n2} + m_{n2}K_{h2}m_{n2} + K_{h2}m_{n2}m_{n2})I + 2(2\alpha^2(m_{n2}m_{n2}K + m_{n2}K_{h2}m_{n2} - m_{n2}K_{h2}m_{n2} - m_{n2}K_{h2}m_{n2}))}
\]

Substituting Eq. (20) into Eqs (18) and (19), one can obtain the temperatures of the hot and cold junction of the two-stage thermoelectric refrigerator.

If the heat transfer irreversibilities are neglected, i.e. \( K_{r} \rightarrow \infty \), Eqs (15) and (20) can be reduced to:

\[
T_{m2} = \frac{0.5m_{n2}I^2R + 0.5m_{n2}I^2R + m_{n2}KT_{m2} + m_{n2}KT_{e2}}{m_{n2}\alpha I - m_{n2}\alpha I + m_{n2}K + m_{n2}K} \quad \ldots(23)
\]
Eq. (25) is a 11 orders polynomial equation which can be rearranged as:

\[ \sum_{i=0}^{11} A_i I^i = 0 \]  

(26)

where \( A_i \) \((i=1-11)\) are the coefficients of the polynomial equation. The coefficients of the equation are functions of the parameters of thermoelectric element pairs \((\alpha, K\) and \(R\)) and parameters concerning heat transfer \((K_{H1}, K_{L1}, K_{H2}, \) and \(K_{L2}\)), so the parameters affect the coefficients of the equation, and then affect the working electrical current and the performance of the device. However, there is no analytical solution for 11 orders polynomial equation. For given parameters, one can obtain the system stable current \(I_s\) from Eq. (26). Substituting \(I_s\) into Eqs (1) and (12), one can obtain the cooling load \(Q_{L2}\) and the COP (coefficient of performance) of the two-stage combined thermoelectric device as:

\[ \text{COP} = Q_{L2} / Q_{H1} \]  

(27)

3 Numerical Optimization and Analysis

3.1 Design variables and fixed parameters

In order to describe the allocations of thermoelectric elements, three ratios of thermoelectric elements are defined: the total thermoelectric element ratio \(x = (m_h + m_e) / M\), i.e. the ratio of the thermoelectric elements of the two-stage thermoelectric generator to the total thermoelectric elements of the combined irreversible device \((M = m_h + m_e + m_{2h} + m_{2e})\); the two-stage thermoelectric generator thermoelectric element ratio \(x_1 = m_h / (m_h + m_e)\), i.e. the ratio of the thermoelectric elements of the top stage of the two-stage thermoelectric generator to the total thermoelectric elements of the two-stage thermoelectric generator; and the two-stage thermoelectric refrigerator thermoelectric element ratio \(x_2 = m_{2h} / (m_{2h} + m_{2e})\), i.e. the ratio of the thermoelectric elements of the top stage of the two-stage thermoelectric refrigerator to the total thermoelectric elements of the two-stage thermoelectric refrigerator. Then, one has:

\[ m_h = x_1 M, \quad m_e = x(1 - x_1) M, \quad m_{2h} = (1 - x)x_2 M \quad \text{and} \quad m_{2e} = (1 - x)(1 - x_2) M. \]

In order to describe the allocations of the thermal conductance, three ratios of thermal conductance are defined: the total thermal conductance ratio \(f = (K_{H1} + K_{L1}) / K_T\), i.e. the ratio of the thermal conductance of the two-stage thermoelectric generator to the total thermal conductance of the combined irreversible device \((K_f = K_{H1} + K_{L1} + K_{H2} + K_{L2})\); the two-stage thermoelectric generator thermal conductance ratio \(f_1 = K_{H1} / (K_{H1} + K_{L1})\), i.e. the ratio of the thermal conductance of the high-temperature side heat exchanger of the two-stage thermoelectric generator to the total thermal conductance of the two-stage thermoelectric generator; and the two-stage thermoelectric refrigerator thermal conductance ratio \(f_2 = K_{H2} / (K_{H2} + K_{L2})\), i.e. the ratio of the thermal conductance of the high-temperature side heat exchanger of the two-stage thermoelectric refrigerator to the total thermal conductance of the two-stage thermoelectric refrigerator. Then, one has:

\[ K_{H1} = f f_1 K_T, \quad K_{L1} = f(1 - f_1) K_T, \quad K_{H2} = (1 - f)f_2 K_T \quad \text{and} \quad K_{L2} = (1 - f)(1 - f_2) K_T. \]

It is obvious that for given source temperature \(T_{H1}\), \(T_{L1}\), \(T_{H2}\) \(T_{L2}\) and figure of merit of thermoelectric elements \(\alpha, K\), and \(R\), the working electrical current depends on the design variables \(x, x_1, x_2, f, f_1, \) and \(f_2\). For given parameters, electrical current solution \(I_s\) can be obtained by numerical calculation from Eq. (25). Substituting the electrical current into Eqs (12) and (27) one has the cooling load \(Q_{L2}\) and COP. The problem is how to determine \(x, x_1, x_2, f, f_1, \) and \(f_2\), so that the maximum values of cooling load and COP are arrived.
For maximum cooling load, the design variables \( x, x_1, x_2, f, f_1 \) and \( f_2 \) should satisfy:

\[
\frac{\partial Q_{l2}}{\partial x} = \frac{\partial Q_{l2}}{\partial x_1} = \frac{\partial Q_{l2}}{\partial x_2} = \frac{\partial Q_{l2}}{\partial f} = \frac{\partial Q_{l2}}{\partial f_1} = \frac{\partial Q_{l2}}{\partial f_2} = 0 \quad \ldots(28)
\]

For maximum COP, the design variables \( x, x_1, x_2, f, f_1 \) and \( f_2 \) should satisfy:

\[
\frac{\partial \text{COP}}{\partial x} = \frac{\partial \text{COP}}{\partial x_1} = \frac{\partial \text{COP}}{\partial x_2} = \frac{\partial \text{COP}}{\partial f} = \frac{\partial \text{COP}}{\partial f_1} = \frac{\partial \text{COP}}{\partial f_2} = 0 \quad \ldots(29)
\]

The objective functions \( Q_{l2} \) and COP are 6-variables functions. However, the electrical current in Eq. (26) is a high-ordered polynomial equation. There is no analytical solution if the equation order is bigger than 4. So the explicit solution of Eqs (26), (28) and (29) cannot be obtained theoretically. The problem needs to be translated. Take the electrical current Eq. (26) as a equality constraint, then the problem is translated into: to determine \( x, x_1, x_2, f, f_1, f_2 \) and \( I \), so that the 7-variables objective function satisfying non-linear equality constraint in Eq. (26), reaches the maximum value. This problem can be solved by Matlab Optimization Toolbox.

The objective functions are set as follows:

\[
\text{max } Q_{l2}(x, x_1, x_2, f, f_1, f_2, I) \quad \ldots(30)
\]

\[
\text{max } \text{COP}(x, x_1, x_2, f, f_1, f_2, I) \quad \ldots(31)
\]

The lower bounds and upper bounds are:

\[
lb = [0 \ 0 \ 0 \ 0 \ 0 \ 0] \quad \ldots(32)
\]

\[
ub = [1 \ 1 \ 1 \ 1 \ 1 \ \inf] \quad \ldots(33)
\]

The non-linear inequalities are:

\[
\begin{bmatrix}
Q_{H1} \\
Q_{l1} \\
Q_{H2} \\
Q_{l2}
\end{bmatrix} > \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix} \quad \ldots(34)
\]

The non-linear equality constraint is Eq. (26).

In the calculations, fixed parameters\(^{10}\) are set as \( \alpha = 4 \times 10^{-4} \text{V/K} \), \( K = 0.03 \text{W/K} \), \( R = 0.002 \Omega \), \( T_{l1} = 300 \text{K} \), and \( T_{H2} = 300 \text{K} \). The optimization progress may be terminated at local optimal point. In quest of global optimal point, several initial values should be tried. When the parameters change, optimized optimum variables obtained last time could be chosen as initial value this time. At a work conditions of \( T_{H1} = 450 \text{K} \), \( T_{l1} = 300 \text{K} \), \( T_{H2} = 300 \text{K} \), and \( T_{l2} = 280 \text{K} \), the optimal variables for maximum cooling load are \( x = 0.6322, x_1 = 0.4945, x_2 = 0.5590 \), \( f = 0.6063, f_1 = 0.5054, f_2 = 0.5484 \) and \( I = 7.8191 \); those for maximum COP are \( x = 0.5009, x_1 = 0.4887, x_2 = 0.5554 \), \( f = 0.4783, f_1 = 0.4943, f_2 = 0.5508 \) and \( I = 5.8262 \). These can be chosen as initial values for other work conditions.

### 3.2 Effect of two-stage thermoelectric generator heat source temperature

Table 1 presents the effect of the two-stage thermoelectric generator heat source temperature on the design variables and the corresponding performances for maximum cooling load. In the optimization, the two-stage thermoelectric refrigerator is working at a temperature difference of 20K \( (T_{l2} = 280 \text{K}) \). The superscript \( Q \) represents optimal variables corresponding to the maximum cooling load \( Q_{l2, \text{max}} \). \( Q_{l2, \text{max}} \) is the optimized cooling load. For comparison, the non-optimized cooling load \( Q_{l2} \) which is calculated at \( x = x_1 = x_2 = f = f_1 = f_2 = 0.5 \) is also presented in Table 1. It can be seen that the two-

| \( T_{H1}(\text{K}) \) | \( x^O \) | \( x_1^O \) | \( x_2^O \) | \( f^O \) | \( f_1^O \) | \( f_2^O \) | \( I^O(\text{A}) \) | \( Q_{l2, \text{max}}(\text{W}) \) | Improvement %
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<td>13.1765</td>
<td>37.4409</td>
<td>35.776</td>
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two-stage thermoelectric generator heat source temperature has large effect on $x^0$ and $f^0$, small effect on $x^0_1$, $x^0_2$, $f^1_1$, and $f^2_2$. With the increase of two-stage thermoelectric generator heat source temperature $T_{H1}$, variables $x^0_1$, $f^1_1$ and $f^2_2$ increase, that means more thermoelectric elements should be allocated to the high temperature side of the two-stage thermoelectric refrigerator and more thermal conductance should be allocated to the high temperature side of the two-stage thermoelectric generator and the high temperature side of the two-stage thermoelectric refrigerator; variables $x^0_2$, and $f^0$ decrease, that means thermoelectric elements allocated to the two-stage thermoelectric generator and the high temperature side of the two-stage thermoelectric generator should be reduced, and the thermal conductance allocated to the two-stage thermoelectric refrigerator should be reduced.

Table 2 presents the effect of two-stage thermoelectric generator heat source temperature on the design variables and the corresponding performances for maximum COP. In the optimization, the two-stage thermoelectric refrigerator is working at a temperature difference of 20 K ($T_{L2} = 280$ K). The superscript COP represents optimal variables corresponding to the maximum COP ($COP_{max}$). $COP_{max}$ is the optimized COP. For comparison, the non-optimized COP which is calculated at $x = x_1 = x_2 = f = f_1 = f_2 = 0.5$ is also presented in Table 2. It can be seen that the two-stage thermoelectric generator heat source temperature has large effect on $x^{COP}_1$ and $f^{COP}_1$, small effect on $x^{COP}_2$, $f^{COP}_2$, and $f^{COP}_2$. With the increase of two-stage thermoelectric generator heat source temperature $T_{H1}$, variables $x^{COP}_2$ and $f^{COP}_2$ increase after their minimum value at about $T_{H1} = 440$ K, that is, more thermoelectric elements and thermal conductance should be allocated to the high temperature side of the two-stage thermoelectric refrigerator; variables $x^{COP}_1$, $x^{COP}_2$, $f^{COP}_1$, and $f^{COP}_2$ decrease, which means the thermoelectric elements and thermal conductance allocated to two-stage thermoelectric generator and the high temperature side of the two-stage thermoelectric generator should be reduced.

Figure 2 shows the performance limits of the combined device i.e. the maximum cooling load ($Q_{L2,max}$) and the maximum COP ($COP_{max}$) versus the two-stage thermoelectric generator heat source temperature $T_{H1}$ with the corresponding optimal variables, respectively. For comparison, the non-optimized cooling load and COP are also plotted in the Fig. 2 by dotted lines. The non-optimized cooling load and COP are calculated at $x = x_1 = x_2 = f = f_1 = f_2 = 0.5$. It can be seen that there is large difference among the optimized cooling load and COP and these non-optimized. By optimization, the cooling load and COP are improved much (the improvements are listed in Tables 1 and 2). Therefore, the variables optimization is necessary and effective for different two-stage thermoelectric generator heat source temperature. The maximum cooling load

Table 2 — Effect of two-stage thermoelectric generator heat source temperature on the design variables and the corresponding performances for maximum COP

<table>
<thead>
<tr>
<th>$T_{H1} (K)$</th>
<th>$x^{COP}_1$</th>
<th>$x^{COP}_2$</th>
<th>$f^{COP}_1$</th>
<th>$f^{COP}_2$</th>
<th>$f^{COP}_1$</th>
<th>$f^{COP}_2$</th>
<th>$COP_{max}$</th>
<th>COP Improvement</th>
</tr>
</thead>
<tbody>
<tr>
<td>400</td>
<td>0.6227</td>
<td>0.4928</td>
<td>0.5586</td>
<td>0.5972</td>
<td>0.4962</td>
<td>0.5547</td>
<td>4.9003</td>
<td>0.0549</td>
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<tr>
<td>440</td>
<td>0.5207</td>
<td>0.4895</td>
<td>0.5557</td>
<td>0.4975</td>
<td>0.4932</td>
<td>0.5511</td>
<td>5.6635</td>
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<tr>
<td>480</td>
<td>0.4502</td>
<td>0.4859</td>
<td>0.5551</td>
<td>0.4295</td>
<td>0.4990</td>
<td>0.5500</td>
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<td>520</td>
<td>0.3978</td>
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<td>0.5498</td>
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<tr>
<td>560</td>
<td>0.3570</td>
<td>0.4785</td>
<td>0.5555</td>
<td>0.3415</td>
<td>0.4828</td>
<td>0.5498</td>
<td>7.1338</td>
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<td>600</td>
<td>0.3243</td>
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<td>0.4788</td>
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<td>7.4660</td>
<td>0.1445</td>
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</table>

Fig. 2 — Maximum cooling load and the maximum COP versus the two-stage thermoelectric generator heat source temperature
and the maximum coefficient COP increase with the increase of two-stage thermoelectric generator heat source temperature $T_{H1}$. The non-optimized COP decreases if the two-stage thermoelectric generator heat source temperature $T_{H1} > 560$ K which is unexpected.

Comparing Table 1 with Table 2, one can see that the optimal variables $x^0$, $x_3^0$, $x_2^0$, $f^0$, $f_1^0$, and $f_2^0$ at maximum cooling load are larger than these $x^{\text{COP}}$, $x_1^{\text{COP}}$, $x_2^{\text{COP}}$, $f^{\text{COP}}$, $f_1^{\text{COP}}$, and $f_2^{\text{COP}}$ at maximum COP. This means that in the optimal range of design variables between maximum cooling load and maximum COP, larger variables lead larger cooling load, while smaller variables lead larger COP.

From the point of view of finite time thermodynamic optimization (the compromise optimization between cooling load and the COP), the optimal variables and working electrical current should satisfy:

$$
\begin{bmatrix}
x^{\text{COP}} \\
x_1^{\text{COP}} \\
x_2^{\text{COP}} \\
f^{\text{COP}} \\
f_1^{\text{COP}} \\
f_2^{\text{COP}} \\
f^{0}
\end{bmatrix} < 
\begin{bmatrix}
x \\
x_1 \\
x_2 \\
f \\
f_1 \\
f_2 \\
f^{0}
\end{bmatrix} \text{ ... (35)}
$$

As a consequence, the design performance optimal range of the thermoelectric device should satisfy consequently:

$$
\begin{bmatrix}
Q_{\text{L2, max}}^{\text{COP}} \\
\text{COP}^{0}
\end{bmatrix} < 
\begin{bmatrix}
Q_{\text{L2}} \\
\text{COP}
\end{bmatrix} < 
\begin{bmatrix}
Q_{\text{L2, max}} \\
\text{COP}_\text{max}
\end{bmatrix} \text{ ... (36)}
$$

Figures 3-5 show the optimal ranges of working electrical currents, cooling load and COP versus the two-stage thermoelectric generator heat source temperature. It can be seen that $I^{0}$, $I^{\text{COP}}$, $Q_{\text{L2, max}}$, $Q^{\text{COP}}$, $\text{COP}_\text{max}$, and COP increase with the increase of the two-stage thermoelectric generator heat source temperature. The differences, i.e. the optimal ranges between the working electric current, cooling load, COP for maximum cooling load and these for maximum COP increase with the increase of two-stage thermoelectric generator heat source.
temperature. The ranges of the optimal variables $x$, $x_1$, $x_2$, $f$, $f_1$ and $f_2$ can be seen from the comparison between Tables 1 and 2.

### 3.3 Effect of two-stage thermoelectric refrigerator cooling space temperature

Table 3 presents the effect of two-stage thermoelectric refrigerator cooling space temperature on the design variables and the corresponding performances for maximum cooling load. In the optimization, the two-stage thermoelectric generator is working at a temperature difference of $150 \, \text{K}$ ($T_{H1} = 450 \, \text{K}$). It can be seen that the two-stage thermoelectric refrigerator cooling space temperature has large effect on $x^0$, $x_2^0$, $f^0$, and $f_2^0$, small effect on $x_1^0$ and $f_1^0$. With the increase of two-stage thermoelectric refrigerator cooling space temperature $T_{L2}$, variables $x^0$, $f^0$ increase which means more thermoelectric elements and thermal conductance should be allocated to the high temperature side of the two-stage thermoelectric generator (the changes are so small that they can be neglected). Variables $x_2^0$, $f_2^0$ decrease, which means thermoelectric elements and thermal conductance allocated to the two-stage thermoelectric generator and the low temperature side of the two-stage thermoelectric refrigerator should be reduced.

Table 4 presents the effect of two-stage thermoelectric refrigerator cooling space temperature on the design variables and the corresponding performances for maximum COP. In the optimization, the two-stage thermoelectric refrigerator cooling space temperature has large effect on $x^{\text{COP}}$, $x_2^{\text{COP}}$, $f^{\text{COP}}$, and $f_2^{\text{COP}}$, small effect on $x_1^{\text{COP}}$ and $f_1^{\text{COP}}$. With the increase of two-stage thermoelectric refrigerator cooling space temperature $T_{L2}$, all of the variables decrease generally, which means thermoelectric elements and thermal conductance allocated to the two-stage thermoelectric generator, the high temperature side of the two-stage thermoelectric generator and the high temperature side of the two-stage thermoelectric refrigerator should be reduced.

Figure 6 shows the performance limits of the combined device, i.e. the maximum cooling load ($Q_{L2, \text{max}}$) and the maximum COP ($COP_{\text{max}}$) versus the two-stage thermoelectric refrigerator cooling space temperature with the optimal variables. For comparison, the non-optimized cooling load $Q_{L2}$ and COP are also plotted in Fig. 6 by dotted lines. The non-optimized performances are calculated at $x = x_1 = x_2 = f = f_1 = f_2 = 0.5$. It can be seen that there is large difference between the optimized cooling load and COP and these non-optimized. By optimization, the cooling load and COP are improved much. The variables optimization is necessary and effective for

<table>
<thead>
<tr>
<th>$T_{L2}(K)$</th>
<th>$x^{\text{COP}}$</th>
<th>$x_1^{\text{COP}}$</th>
<th>$x_2^{\text{COP}}$</th>
<th>$f^{\text{COP}}$</th>
<th>$f_1^{\text{COP}}$</th>
<th>$f_2^{\text{COP}}$</th>
<th>$f_2^{\text{COP}}$</th>
<th>$Q_{L2, \text{max}}(W)$</th>
<th>$Q_{L2}(W)$</th>
<th>Improvement %</th>
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<th>$T_{L2}(K)$</th>
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<th>$x_1^{\text{COP}}$</th>
<th>$x_2^{\text{COP}}$</th>
<th>$f^{\text{COP}}$</th>
<th>$f_1^{\text{COP}}$</th>
<th>$f_2^{\text{COP}}$</th>
<th>$f_2^{\text{COP}}$</th>
<th>$Q_{L2, \text{max}}(W)$</th>
<th>$Q_{L2}(W)$</th>
<th>Improvement %</th>
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<td>0.1852</td>
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different cooling space temperature. It is important to note that the maximum cooling load $Q_{L2,max}$ and the maximum coefficient COP$_{max}$ increase non-linearly while these non-optimized linearly with the increase of two-stage thermoelectric refrigerator cooling space temperature $T_{L2}$. If the device is non-optimized, the cooling load and the COP fall down to zero when the two-stage thermoelectric refrigerator cooling space temperature $T_{L2} < 267$ K. The extreme cooling temperature difference is about $\Delta T_{2,max} = 200$ K – 267 K = 33 K. By optimization, the extreme cooling temperature difference is about $\Delta T_{2,max} = 300$ K – 240 K = 60 K > 33 K which is considerable in a wider application.

Figures 7-9 show the optimal ranges of working electrical currents, cooling load and COP versus the two-stage thermoelectric refrigerator cooling space temperature. It can be seen that $I^Q$ and $I^{COP}$ decrease while $Q_{L2,max}$, $Q^{COP}$, COP$_{max}$ and COP$^Q$ increase with the increase of the two-stage thermoelectric generator heat source temperature. The differences, i.e. the optimal ranges among the working electric current, cooling load, COP for maximum cooling load and these for maximum COP increase with the increase of two-stage thermoelectric refrigerator cooling space temperature. The ranges of the optimal variables $x$, $x_1$, $x_2$, $f_1$ and $f_2$ can be seen from the comparison between Tables 3 and 4.
4 Conclusions

The finite time thermodynamic model of a combined thermoelectric device, two-stage thermoelectric refrigerator driven by two-stage thermoelectric generator with heat transfer irreversibility, is analyzed. The results show that there exist optimum thermoelectric element allocations and optimum thermal conductance allocations among the two thermoelectric generators and the two thermoelectric refrigerators for fixed total number of thermoelectric elements and fixed thermal conductance corresponding to maximum cooling load or maximum COP, respectively. In order to describe the allocations of thermoelectric elements and external thermal conductance, six design variables are defined for optimization. The solution procedure of optimal performance and the corresponding optimal variables are given and the cooling load and COP of the combined thermoelectric device are optimized. Higher cooling load and higher COP which are considerable for applications are reached, by numerical optimization. Moreover, the effects of the heat source temperature of the two-stage thermoelectric generator and the heat source (cooling space) temperature of the two-stage thermoelectric refrigerator on the optimal performance of the combined thermoelectric device are analyzed. In general, the optimum variables at maximum cooling load are larger than these at maximum COP. Variables optimization should be considered in the design and application of practical combined thermoelectric devices in order to obtain the maximum economy benefit. The results obtained herein may provide guidelines for the design and application of practical combined thermoelectric devices.

Acknowledgement

This paper is supported by The National Natural Science Foundation of P. R. China (Project No. 10905093), Program for New Century Excellent Talents in University of P. R. China (Project No. NCET-04-1006) and The Foundation for the Author of National Excellent Doctoral Dissertation of P. R. China (Project No. 200136).

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<td>M</td>
<td>Total number of thermoelectric elements of the device</td>
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<tr>
<td>m</td>
<td>Number of thermoelectric elements</td>
</tr>
<tr>
<td>T</td>
<td>Temperature (K)</td>
</tr>
<tr>
<td>Q</td>
<td>Rate of heat flow (W)</td>
</tr>
<tr>
<td>R</td>
<td>Total internal electrical resistance of a thermoelectric element (Ω)</td>
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<tr>
<td>x</td>
<td>Ratio of numbers of thermoelectric elements</td>
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Greek letters

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<td>α</td>
<td>Seebeck coefficient of thermoelectric element</td>
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Subscripts

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<td>High temperature side</td>
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<td>h</td>
<td>Hot junction</td>
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Superscripts

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<td>COP</td>
<td>For maximum coefficient of performance</td>
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References

44 Meng F, Chen L & Sun F, Cryogenics, 49 (2009) 57.