Large deflection analysis of rhombic sandwich plates placed on elastic foundation

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This paper presents non-linear analysis of rhombic sandwich plates placed on elastic foundation under uniform load. Banerjee’s hypothesis involving a new form of energy expression in the total potential energy of the system has been employed. As a consequence the differential equation is decoupled keeping intact its non-linear character. The aim of the present study is to analyze the non-linear behaviour of rhombic sandwich plates placed on elastic foundation for different skew angles. The results have been obtained both for movable and immovable edges from a single cubic equation. Numerical results (central deflection versus load) have been computed and compared with known results (K=0) for square sandwich plates only. The corresponding linear analysis (K=0) is also presented. The results thus obtained are in good agreement with those reported in the literature.

Non-linear analysis of thin elastic plates resting on elastic foundation is interesting to design engineers for their wide applications in the practical field. Chien and Chen2 successfully carried out the analysis on the effect of initial stresses of nonlinear vibration of laminated plates on an elastic foundation. Nonlinear partial differential equations based on Mindlin plate theory are derived for nonlinear vibration of laminated plates. Civalek3 has utilized harmonic differential quadrature-finite differences coupled approaches for geometrically nonlinear static and dynamic analyses of rectangular plates on elastic foundation. Winkler-Pasternak foundation model has been considered. This analysis is attractive for the practical field.

Sladek et al.4 reported meshless local boundary integral equation method for simply supported and clamped plates resting on elastic foundation. Simply supported and clamped thin elastic plates resting on a two parameter foundation were analyzed. The governing partial differential equation of fourth order for a plate is decomposed into two coupled partial differential equations of second order. One of them is Poisson’s equation and the other is Helmholtz’s equation.

In the literature on nonlinear analysis of elastic plates the following two methods are attractive:

1. In the generalized differential quadrature method the global Lagrange interpolation polynomial is used, viz.:

\[ g_j(x) = \prod_{k=1,k \neq j}^{N} \frac{x - x_k}{x_j - x_k}, j = 1, 2, ..., N \]

whereas, singular convolution method the following integral equations having a singular convolution is defined as

\[ F(t) = (T * \eta)(t) = \int_{-\infty}^{\infty} T(t-x)\eta(x)dx \]

where, \( T(t-x) \) is a singular kernel. The discrete singular convolution algorithm can be realized by using many approximation kernels.

It is well known that a good number of structural design utilizes sandwich type construction in the fabrication of major structural components. A high strength-to-weight ratio is achieved by combining a relatively thick light-weight core with two thin-high strength faces. The problem of large deflection of sandwich plates has been investigated by several researchers, among which works of Reissner7, Wang8, Hoff9 and Eringen10 need special mention. Reissner presented an exact analysis of finite deflection of sandwich plates. Wang8 gave a general theory of large deflections of sandwich plates and shells. Hoff9 and

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Eringen each developed a theory of bending and buckling of sandwich plates. All these investigations are, however, confined to rectangular sandwich plates under mechanical loading only. Kamiya employed Berger’s well known technique to solve nonlinear problems of sandwich plates using a new set of governing equations with a correction factor. This work has been restricted to a particular plate geometry.

Dutta and Banerjee offered a simple approach to investigate nonlinear static as well as dynamic behaviours of sandwich plates. Later, Ray et al. quite elegantly investigated in the nonlinear thermal behaviour of sandwich plates.

Dumir and Dumir & Bhaskar worked on nonlinear analysis of rectangular plates on elastic foundations. Civalek presented geometrically nonlinear analysis of shells.

Some more interesting works on sandwich structure have been reported where nonlinear analyses have been carried out elegantly and are attractive to design engineers.

Literature on large deflection analysis of elastic skew sandwich plates demands special attention because of their wide applications in space industry.

Plates placed on elastic foundation are also attractive to design engineers for their wide applications. Civalek and Yavas have carried out analysis on large deflection static analysis of rectangular plates. Non-linear analysis of sandwich plates on elastic foundation is rarely reported.

The aim of the present study is to use a set of uncoupled differential equations in oblique coordinates to analyze nonlinear behaviours of rhombic sandwich plates placed on elastic foundation under uniform loading using Banerjee’s Hypothesis. To obtain the central deflection \( \frac{w_0}{h} \) versus load \( \frac{q a d^4}{E h^3} \) Galerkin technique has been used. Numerical results thus obtained for different skew angles have been plotted in graphs. Results for linear and nonlinear analysis have been compared with the results obtained by Chakraborthy and Dutta & Banerjee for square sandwich plates \( (K=0) \) only. The numerical results of the non-linear behaviours for different skew angles of rhombic sandwich plates placed on elastic foundation are believed to be new.

**Governing Equation**

We consider a rhombic sandwich-plate (Fig. 1) with an isotropic core as well as isotropic upper and lower faces of identical thickness; while the faces respond to the bending and membrane action of the plate, the core is assumed to transfer only shear deformation. Moreover, the thickness of upper and lower faces is sufficiently thin in comparison with core thickness \( (h \gg t) \) to ignore a variation of shear in the thickness direction of the faces.

Under mechanical loading the governing equations for sandwich plates on elastic foundation in rectangular Cartesian coordinate are given as:

\[
\begin{bmatrix}
\frac{Et}{2(1-\nu^2)} \nabla^2 - \frac{G}{h} \frac{2Et}{G} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \\
+ h\frac{2}{G(1-\nu^2)} \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \nabla^2 w \\
+ 2 \left( \frac{\partial w}{\partial x} \right)^2 + 2 \left( \frac{\partial w}{\partial y} \right)^2 + 4 \frac{\partial^2 w}{\partial x \partial y} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}
\end{bmatrix}
\]

**Fig. 1 — (a) Sandwich plate and (b) Skew sandwich plate**
\[ \frac{q}{G} \] + \( G \nabla^2 w + kw = 0 \quad \ldots (1) \]

where,

\[ I^m = \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 + \frac{v}{2} \left( \frac{\partial w}{\partial y} \right)^2 \quad \ldots (2) \]

For skew angle \( \theta \), (Fig. 1b) we get the following transformation:

\[ \nabla^2 = \sec^2 \theta \left( \frac{\partial^2}{\partial x_i^2} - 2 \sin \theta \frac{\partial^2}{\partial x_i \partial y_i} + \frac{\partial^2}{\partial y_i^2} \right) \]

\[ \nabla^4 = \sec^4 \theta \left[ \frac{\partial^4}{\partial x_i^4} - 4 \sin \theta \left( \frac{\partial^4}{\partial x_i^2 \partial y_i^2} + \frac{\partial^4}{\partial x_i \partial y_i^4} \right) + 2(1 + 2 \sin^2 \theta) \left( \frac{\partial^4}{\partial x_i^2 \partial y_i^2} + \frac{\partial^4}{\partial y_i^4} \right) \right] \]

Putting the above transformations in Eqs (1) and (2), we get the following set of differential equations:

\[ \left[ \frac{E}{2(1-\nu^2)} \sec^2 \theta \left( \frac{\partial^2}{\partial x_i^2} - 2 \sin \theta \frac{\partial^2}{\partial x_i \partial y_i} + \frac{\partial^2}{\partial y_i^2} \right) - \frac{G}{h} \right] \]

\[ + \tan^2 \theta \frac{\partial^2 w}{\partial y_i^2} + \frac{\partial^2 w}{\partial y_i^2} + h \sec^2 \theta \left( \frac{\partial^2 w}{\partial x_i^2} - 2 \sin \theta \frac{\partial^2 w}{\partial x_i \partial y_i} \right) \]

\[ + \frac{\partial^2 w}{\partial y_i^2} \left[ \frac{E}{G(1-\nu^2)} \left[ \sec \theta \left( \frac{\partial w}{\partial x_i} - \sin \theta \frac{\partial w}{\partial y_i} \right) \right]^2 \right] \]

\[ \left[ \frac{E}{2(1-\nu^2)} \sec^2 \theta \left( \frac{\partial^2 w}{\partial x_i^2} - 2 \sin \theta \frac{\partial^2 w}{\partial x_i \partial y_i} \right) + \frac{q}{G} \right] \]

\[ + \frac{v}{2} \left( \frac{\partial w}{\partial y_i} \right)^2 \quad \ldots (3) \]

\[ I^m = \frac{1}{2} \left( \sec \theta \left( \frac{\partial w}{\partial x_i} - \sin \theta \frac{\partial w}{\partial y_i} \right) \right)^2 + \frac{v}{2} \left( \frac{\partial w}{\partial y_i} \right)^2 \quad \ldots (4) \]

**Analysis**

Let us assume \( w = w_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{a} \quad \ldots (5) \)

\[ q = q_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{a} \quad \ldots (6) \]

This form of \( w \) clearly satisfies the required simply supported edge conditions.

It is to be noted that to determine the desired solution, \( w \) has been chosen in the form of double sine series ensuring convergence of the solution. Moreover, putting this form of \( w \) in the given differential equation, we get the error function

\[ \varepsilon (x, y) = \int w = w_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{a} \]

is not the exact solution of the differential equation. We are now required to minimize this error by using the well known Galerkin’s Technique. This technique gives

\[ \iint_{\varepsilon (x, y)} dx dy = 0 \]
Thus, the solver of the differential equation is Galerkin’s Technique involving the evaluation of the above double integrals, which is simple.

If we now integrate Eq. (4) over the entire plain area of the plate we get

\[ I^m = \frac{1}{8} w_0 \pi^2 \left\{ \sec^2 \theta (1 + \sin^2 \theta) + \nu \right\} \quad \ldots (7) \]

It is to be noted that for movable edge conditions

\[ I^m = 0 \]

Putting Eqs (5), (6) and (7) in Eq. (3), we get the error function, Galerkin’s Technique requires

\[ \int_{x, y} \nu \theta \theta \pi + \pi \theta \theta \sec^8 \frac{\nu}{2} \left( \sin^2 \frac{\nu}{2} \left( 1 + \sin^2 \frac{\nu}{2} \right) + \nu \right\} \quad \ldots (8) \]

Evaluating the integrals in (8) we get the following form of cubic equation, determining the central deflection \( w_0 \);

\[ A \left( \frac{w_0}{h} \right)^3 + B \left( \frac{w_0}{h} \right) + C = 0 \quad \ldots (9) \]

where, \( A = \left[ \frac{\pi^6 E\ell^2}{16 (1 - \nu^2)^3 Ga^2} \right] \left\{ \sec^2 \theta (1 + \sin^2 \theta) + \nu \right\} \), \( B = \left[ \frac{\pi^4 t}{2(1 - \nu^2)h} \left\{ \sec^2 \theta (6 \sec^2 \theta + 3 \tan^2 \theta + 1) + 3 \right\} \right] \), and

\[ C = -\frac{1}{4} \left[ 1 + \frac{\pi^2 Eth \sec^2 \theta}{(1 - \nu^2)Ga^2} \right] q_0 a^4 \frac{Eh^4}{\theta} \]

**Numerical Results**

Tables 1-3 and corresponding graphs (Figs. 2-10) shows the results for the central deflection parameter for different load function of a rhombic sandwich plates on elastic foundation under uniformly

<table>
<thead>
<tr>
<th>( q_0 a^4 )</th>
<th>( \frac{Eh^4}{\theta} )</th>
<th>For ( \theta = 0^\circ )</th>
<th>( \theta = 30^\circ )</th>
<th>( \theta = 45^\circ )</th>
<th>( \theta = 60^\circ )</th>
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<tr>
<th>( q_0 a^4 )</th>
<th>( \frac{Eh^4}{\theta} )</th>
<th>For ( \theta = 0^\circ )</th>
<th>For ( \theta = 30^\circ )</th>
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Table 3 — Non-linear analysis (movable edge conditions $K = 0$)

<table>
<thead>
<tr>
<th>$\frac{q_0 a^4}{Eh^3}$</th>
<th>For $\theta = 0^\circ$</th>
<th>For $\theta = 30^\circ$</th>
<th>For $\theta = 45^\circ$</th>
<th>For $\theta = 60^\circ$</th>
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<td></td>
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<td>$\frac{W_0}{h}$ (known$^{(1)}$)</td>
<td>$\frac{W_0}{h}$ (calculated)</td>
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</table>

Fig. 2 — Linear analysis

Fig. 3 — Nonlinear analysis (immovable edge conditions $K=0$, $K=0.5$, $K=1.0$)

Fig. 4 — Nonlinear analysis (movable edge conditions $K=0$, $K=0.5$, $K=1.0$)

Fig. 5 — Nonlinear analysis (immovable edge conditions $K=2$)

**Conclusions**

The proposed differential equations are uncoupled and thus simple. From the same Cubic Eq. (9) for $\left(\frac{W_0}{h}\right)$, the results of immovable and movable edge conditions can be obtained. Numerical results (both linear and non-linear) for rhombic sandwich plates on elastic foundation are obtained for different
skew angles and presented those in tables. For square sandwich plate the results are compared with those of known results for \(K=0\) and are in good agreement. The results for the other skew angles for different values of \(K\) are believed to be completely new. The numerical results presented in different tables for different skew angles and different values of \(K\), offer an interesting observation. As \(\theta\) increases, i.e., as the plate tends towards rhombic shape, the deflection \(\frac{w_0}{h}\) decreases, i.e., stress decreases. This is quite expected because with the increase of skew angles, the plate offers more rigid structure. And also as the value of \(K\) increases, which means elastic force increases, the deflection \(\frac{w_0}{h}\) decreases, as elastic force always acts in upward direction, thereby opposing the deflection.

Acknowledgement

We are grateful to Dr B Banerjee, Head of the Department of Mathematics, Jalpaiguri Government Engineering College (Retd.), Jalpaiguri, West Bengal, for his valuable suggestions.
**Nomenclature**

- $E$ = Young’s modulus.
- $q, q_0$ = uniform load
- $\nu$ = Poisson’s Ratio
- $a$ = size of plate.
- $t$ = face thickness.
- $h$ = core thickness.
- $G$ = shear modulus.
- $u, v, w$ = displacement in $x, y, z$ directions respectively.
- $\lambda$ = constant depending on the Poisson’s ratio of the plate materials.
- $I^*$ = first invariant of average face strain
- $k$ = Winkler foundation parameter
- $K = \frac{ka^2}{4Eh^3}$ = stiffness parameters of Winkler foundation
- $\theta$ = Skew angle

**References**